How to Visualize a Graph: Specification and Algorithms

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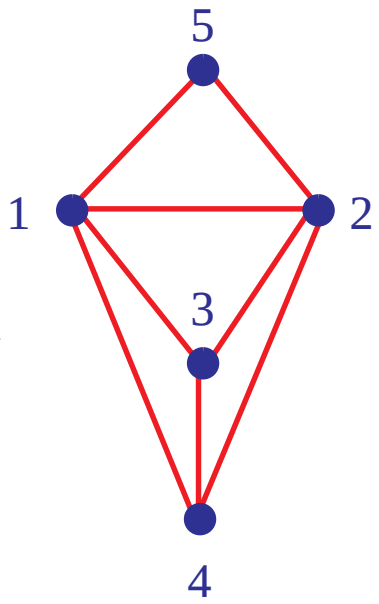
Introduction

Graph Drawing

- geometry → graph
  - intersection graphs
  - visibility graphs
- graph → geometry
  - graph drawing

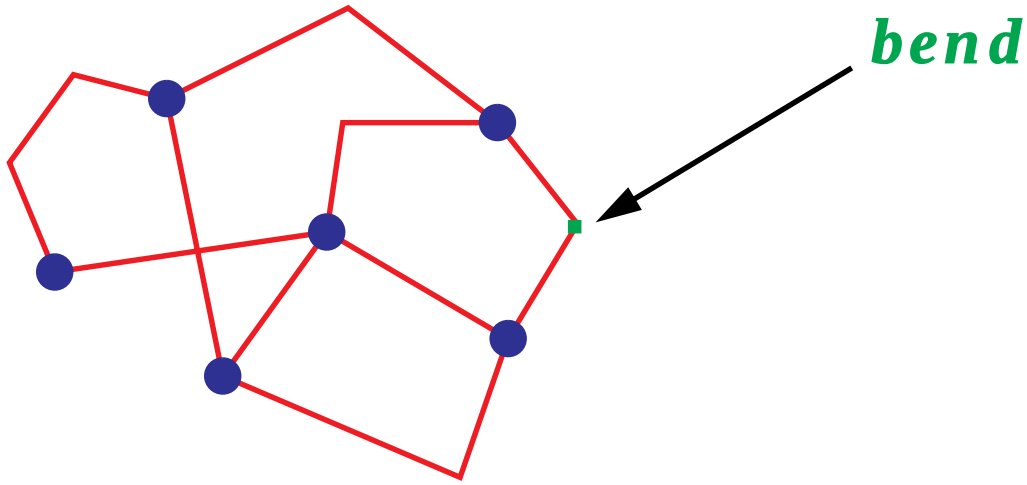
G=(V,E)
V={1,2,3,4,5}
E={(1,2), (1,3), (1,4), (1,5), (2,3), (2,4), (2,5), (3,4)}



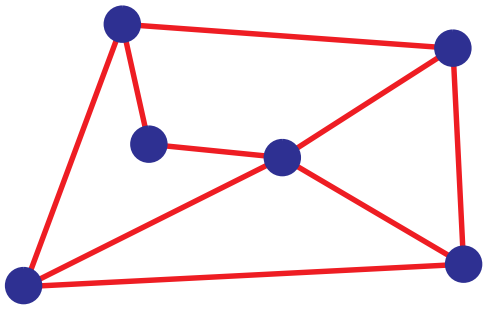
- examples of applications
  - software engineering: subroutine-call graphs, data-flow diagrams, entity-relationship diagrams
  - project management: PERT diagrams
  - knowledge representation
  - circuit schematics

Drawings of Graphs

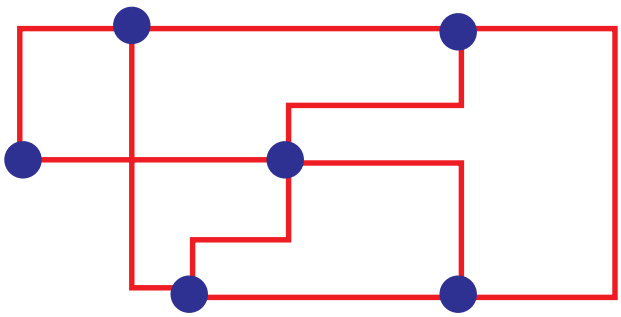
polyline drawing



planar straight-line drawing

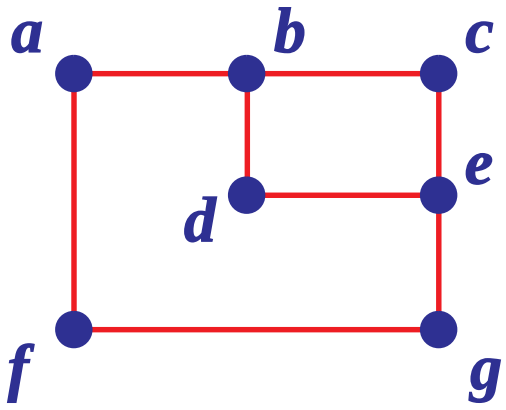


orthogonal drawing

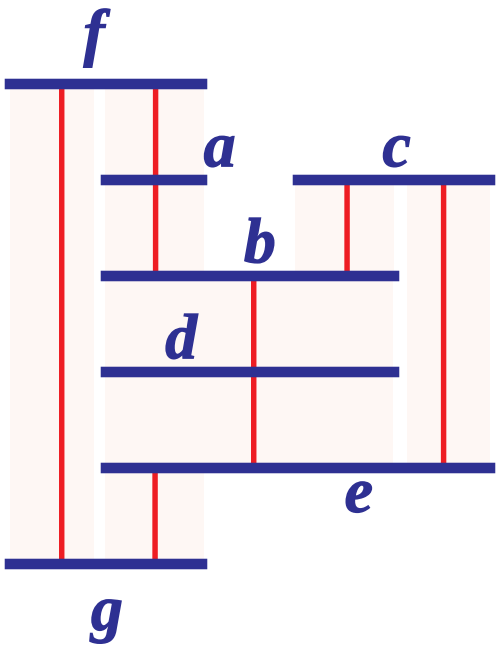


Drawings of Graphs

planar rectilinear drawing

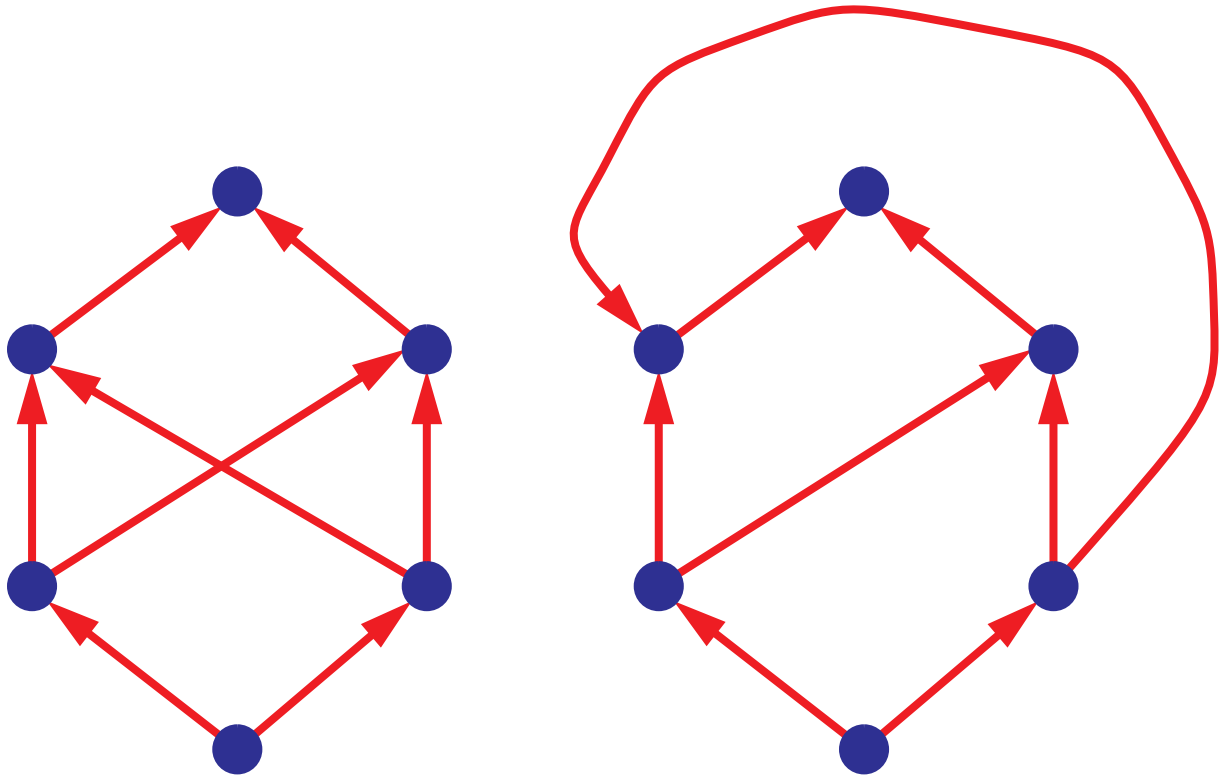


strong visibility representation



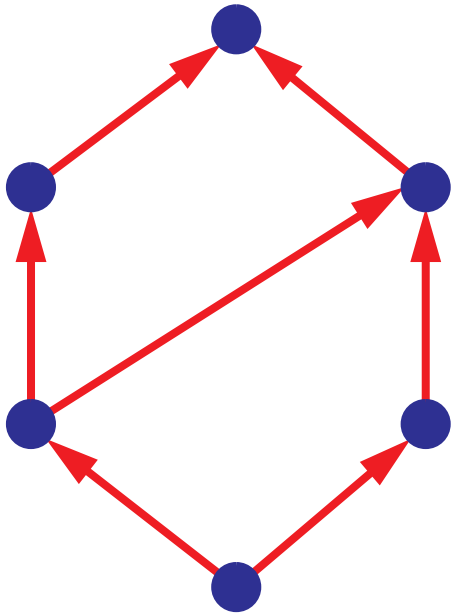
Upward Drawings

- directed acyclic graphs are usually drawn in such a way that all edges “flow” in the same direction, e.g., from left to right, or from bottom to top
- such ***upward*** drawings effectively visualize hierarchical relationships, such as covering digraphs of ordered sets
- not every planar acyclic digraph admits a planar upward drawing

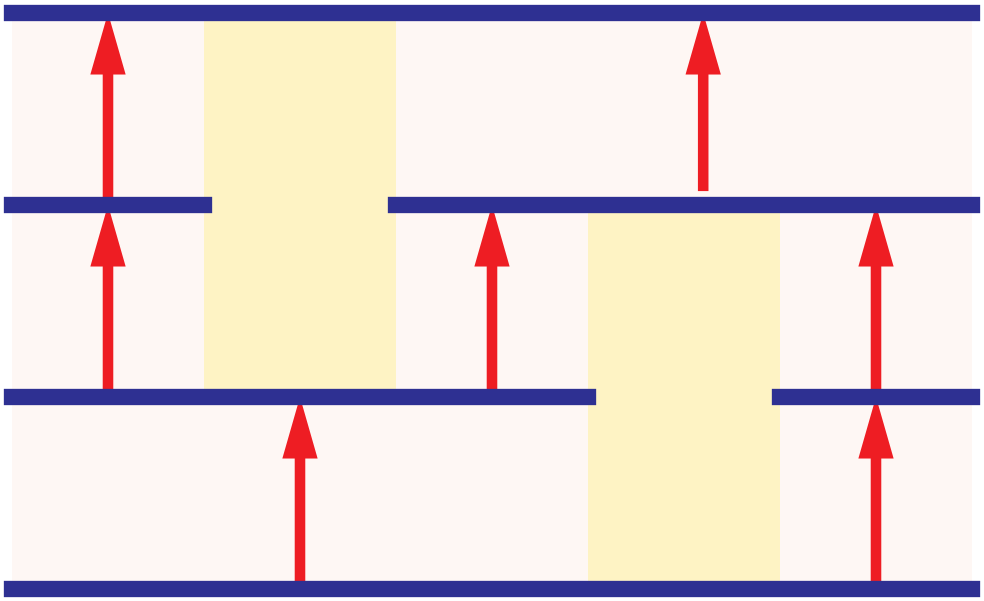


Upward Drawings of Digraphs

upward planar straight-line drawing



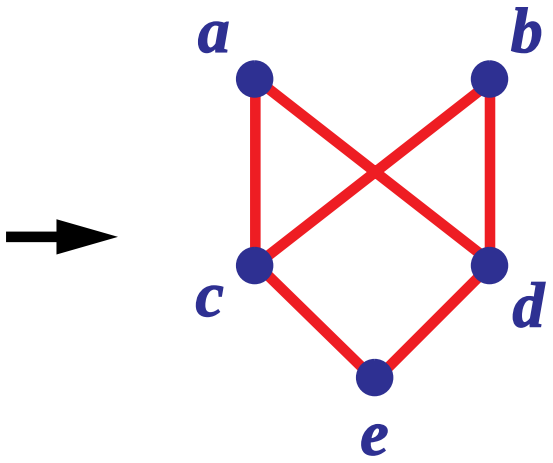
weak visibility representation



How to Draw an Ordered Set

- a “natural” geometric representation of an ordered set is an ***upward drawing*** of its covering digraph, with arrows usually omitted

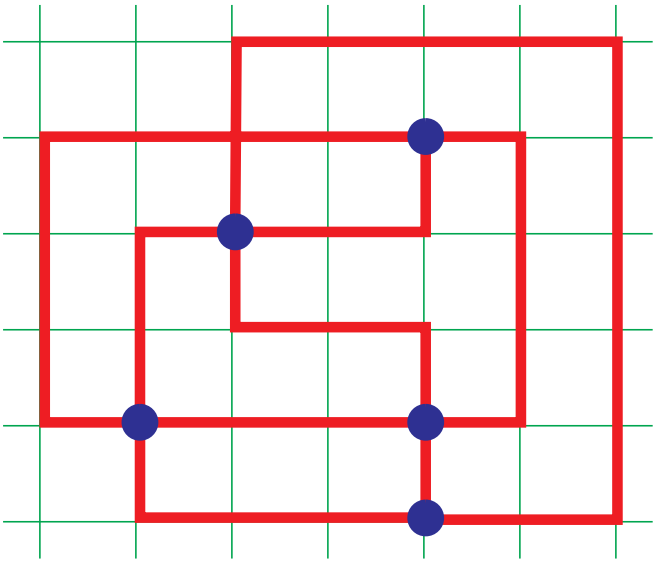
P={(a,c), (a,d), (a,e), (b,c), (b,d), (b,e), (c,e), (d,e)}



- covering digraphs of ordered sets are acyclic digraphs without ***transitive edges***, hence the problem of drawing ordered sets is an important special case of the problem of drawing acyclic digraphs

Resolution

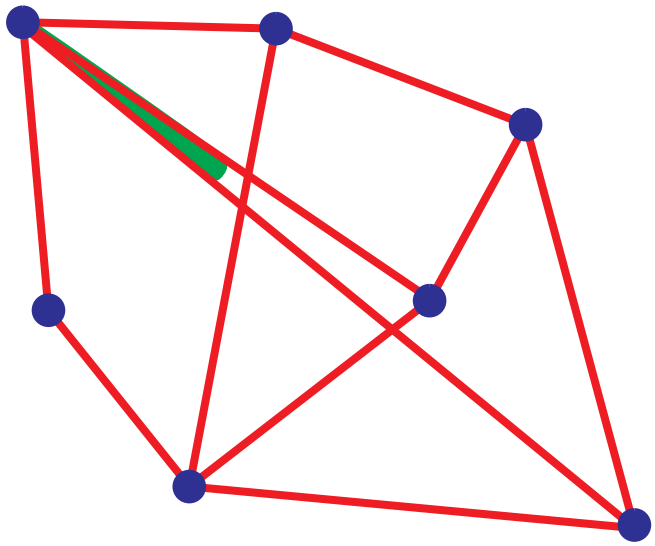
- display devices and the human eye have finite resolution
- examples of ***resolution rules***:
  - integer coordinates for vertices and bends (***grid*** drawings)



  - prescribed minimum distance between vertices
  - prescribed minimum distance between vertices and nonincident edges
  - prescribed minimum angle formed by consecutive incident edges (***angular resolution***)

Angular Resolution

- The ***angular resolution*** ρ of a straight-line drawing is the smallest angle formed by two edges incident on the same vertex



- ***High angular resolution*** is desirable in ***visualization*** applications and in the design of ***optical communication*** networks.
- A ***trivial upper bound*** on the angular resolution is

$$\rho \leq \frac{2\pi}{d}$$

Aesthetic Criteria

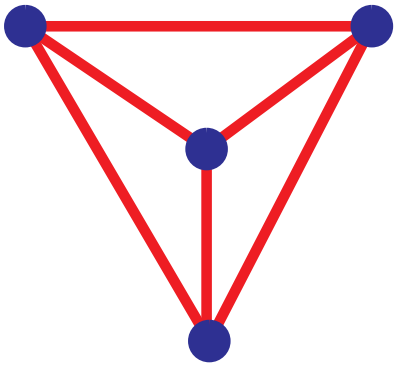
- some drawings are better than others in conveying information on the graph
- ***aesthetic criteria*** attempt to characterize readability by means of general ***optimization*** goals

Examples

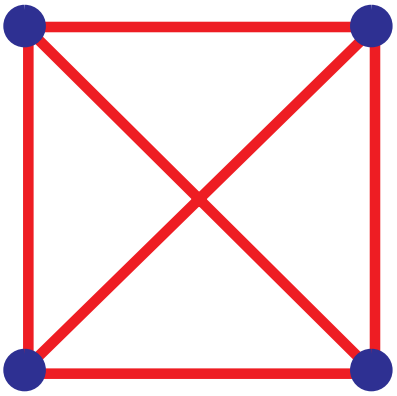
- minimize ***crossings***
- minimize ***area***
- minimize ***bends*** (in orthogonal drawings)
- minimize ***slopes*** (in polyline drawings)
- maximize ***smallest angle***
- maximize display of ***symmetries***

Trade-Offs

- in general, one cannot simultaneously optimize two aesthetic criteria



min # crossings

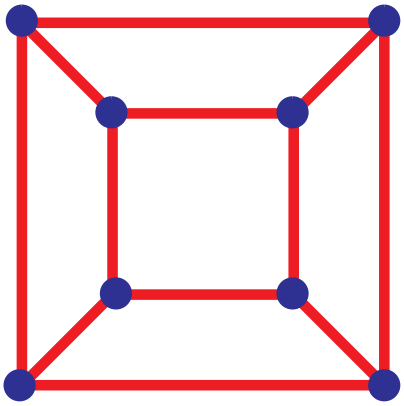


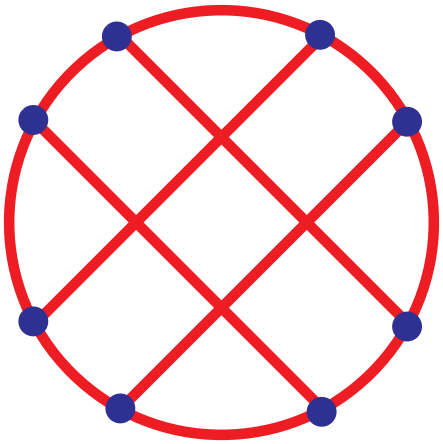
max symmetries

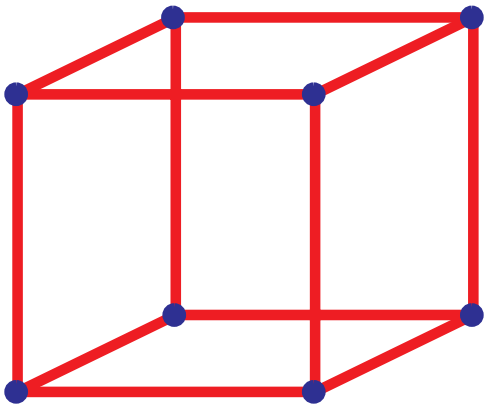
Complexity Issues

- testing planarity takes linear time
- testing upward planarity is NP-hard
- minimizing crossings is NP-hard
- minimizing bends in planar orthogonal drawing:
  - NP-hard in general
  - polynomial time for a fixed embedding

Beyond Aesthetic Criteria







Constraints

- some readability aspects require knowledge about the semantics of the specific graph (e.g., place “most important” vertex in the middle)
- ***constraints*** are provided as additional input to a graph drawing algorithm

Examples

- place a given vertex in the “middle” of the drawing
- place a given vertex on the external boundary of the drawing
- draw a subgraph with a prescribed “shape”
- keep a group of vertices “close” together

Algorithmic Approach

- Layout of the graph generated according to a prespecified set of aesthetic criteria
- Aesthetic criteria embodied in an algorithm as optimization goals. E.g.
  - minimization of crossings
  - minimization of area

Advantages

- Computational ***efficiency***

Disadvantages

- User-defined ***constraints*** are not naturally supported

Extensions

- A limited constraint-satisfaction capability is attainable within the algorithmic approach

  E.g., [Tamassia Di Battista Batini 87]

Declarative Approach

- Layout of the graph specified by a ***user-defined*** set of ***constraints***
- Layout generated by the ***solution*** of a ***system*** of constraints

Advantages

- ***Expressive power***

Disadvantages

- Some natural aesthetics (e.g., planarity) need ***complicated*** constraints to be expressed
- General constraint-solving systems are computationally ***inefficient***
- Lack of a powerful language for the specification of constraints (currently done with a detailed enumeration of facts, or with a set notation)

Getting Started with Graph Drawing

- Annotated bibliography on graph drawing (more than 300 entries) by Di Battista, Eades, Tamassia, and Tollis.

  Computational Geometry: Theory and Applications, 4(5), 235-282 (1994). Preprint:

  `ftp://ftp.cs.brown.edu/pub/papers/compgeo/`

- Computational geometry bibliography (more than 5,000 BibTeX entries, including most papers on graph drawing).

  `ftp://cs.usask.ca/pub/geometry`

  `http://www.cs.ruu.nl/people/otfried/htmlgeombib.html`

- Proceedings of Graph Drawing '93

  `ftp://ftp.cs.brown.edu:pub/papers/compgeo/`

- Proceedings of Graph Drawing '94
  LNCS vol. 894, Springer-Verlag, 1995.

  Order from `order@springer.de`

- Book on graph drawing
  by Di Battista, Eades, Tamassia, and Tollis, to be published by Prentice Hall in 1996.

Publishing with Graph Drawing

- The area of graph drawing has received increasing interest in the last years.
- Special issues on graph drawing, to appear in Algorithmica, CGTA, and JVLC.
- Papers on graph drawing are being presented at many conferences in combinatorics and computer science, including VL, CHI, AVI, STOC, FOCS, SODA, ESA, CG, SPAA, SPDP, WG, ORDAL, WADS, SWAT, ...
- Annual workshop on graph drawing GD 92, 93, 94, 95, ...

How to Visualize a Graph: Specification and Algorithms

Part I: Algorithmic Approaches

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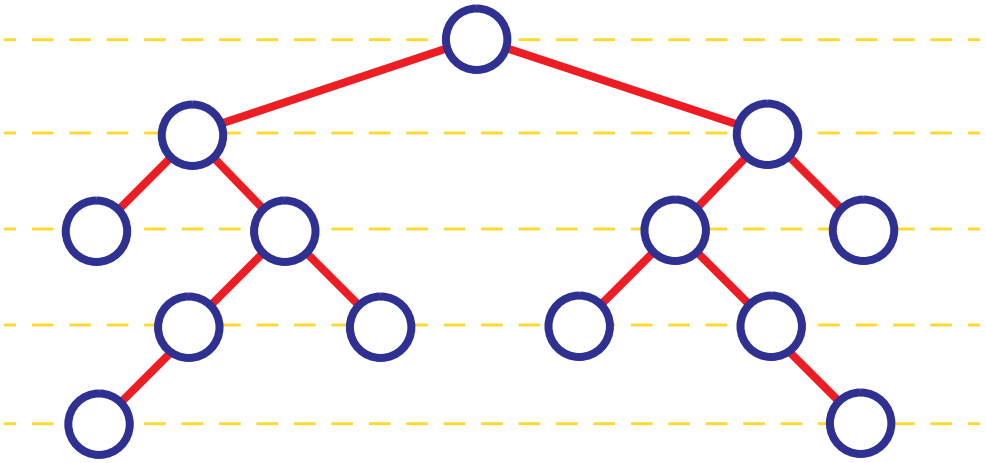
Outline of Part I

- Trees
- Planar Undirected Graphs
- Planar Directed Graphs
- General Undirected Graphs
- General Directed Graphs
- Systems
- Challenges and Open Problems

Trees

Drawings of Rooted Trees

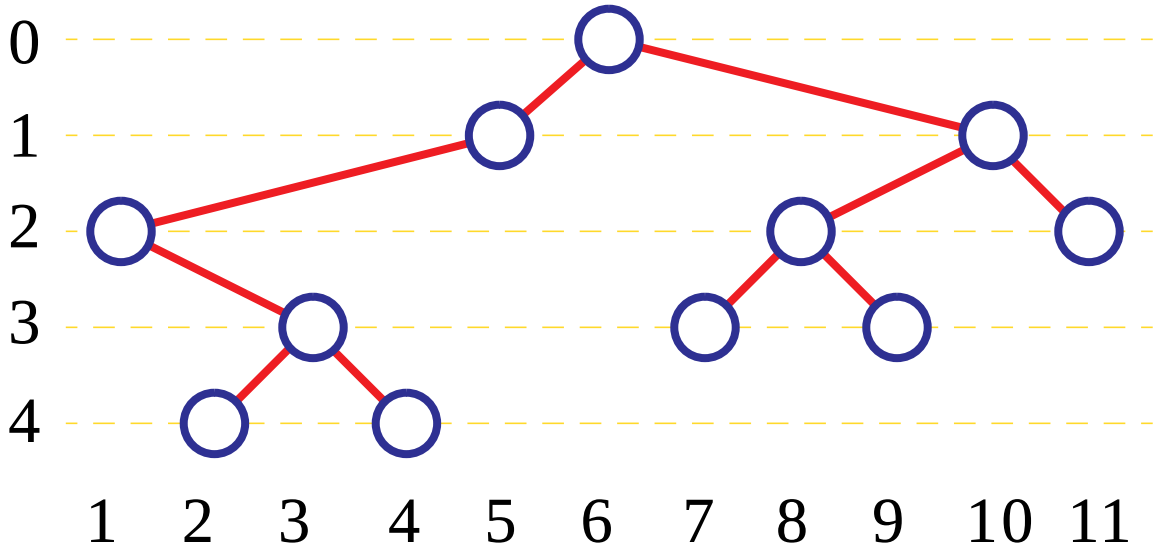
- the usual drawings of rooted trees are ***planar***, ***straight-line***, and ***upward*** (parents above children)
- it is desirable to minimize the ***area*** and to display ***symmetries*** and ***isomorphic subtrees***
- ***level drawing***: nodes at the same distance from the root are horizontally aligned



- level drawings may require $\Omega(n^2)$ ***area***

A Simple Level Drawing Algorithm for Binary Trees

- y(v) = distance from root
- x(v) = inorder rank

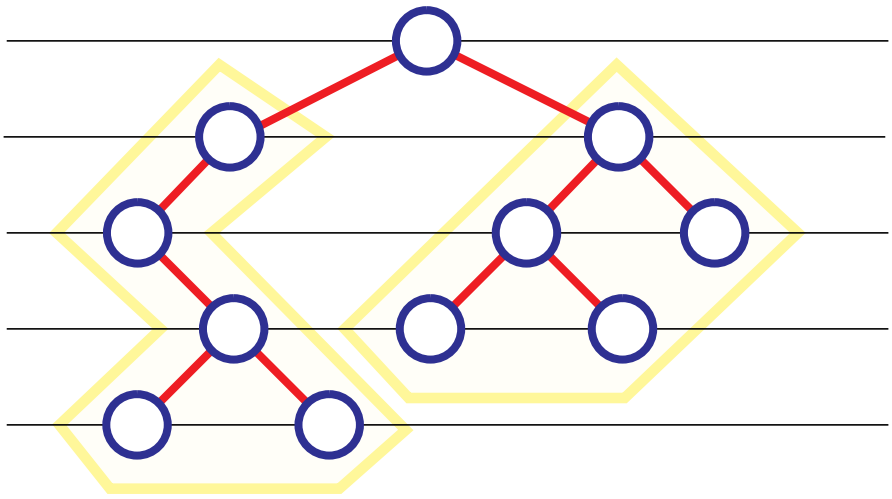


- level grid drawing
- display of symmetries and of isomorphic subtrees
- parent in between left and right child
- parents not always centered on children
- width = n – 1

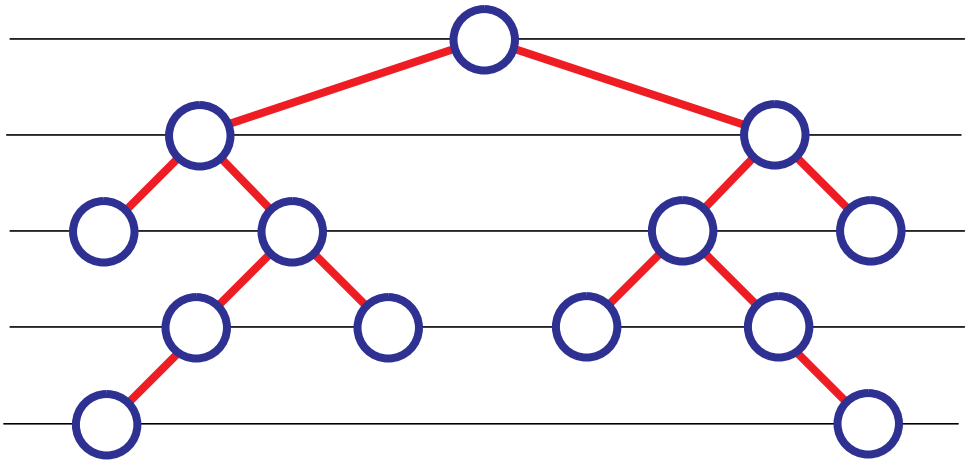
A Recursive Level Drawing Algorithm for Binary Trees

[Reingold Tilford 1983]

- draw the left subtree
- draw the right subtree
- place the drawings of the subtrees at horizontal distance 2
- place the root one level above and half-way between the children
- if there is only one child, place the root at horizontal distance 1 from the child

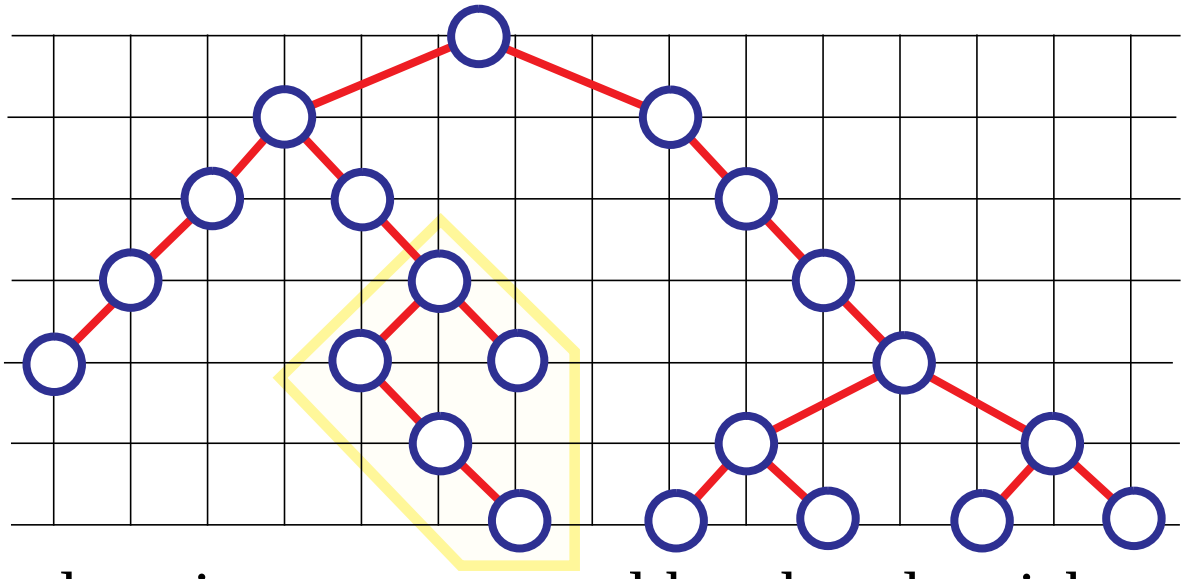


Properties of Recursive Level Drawing Algorithm for Binary Trees

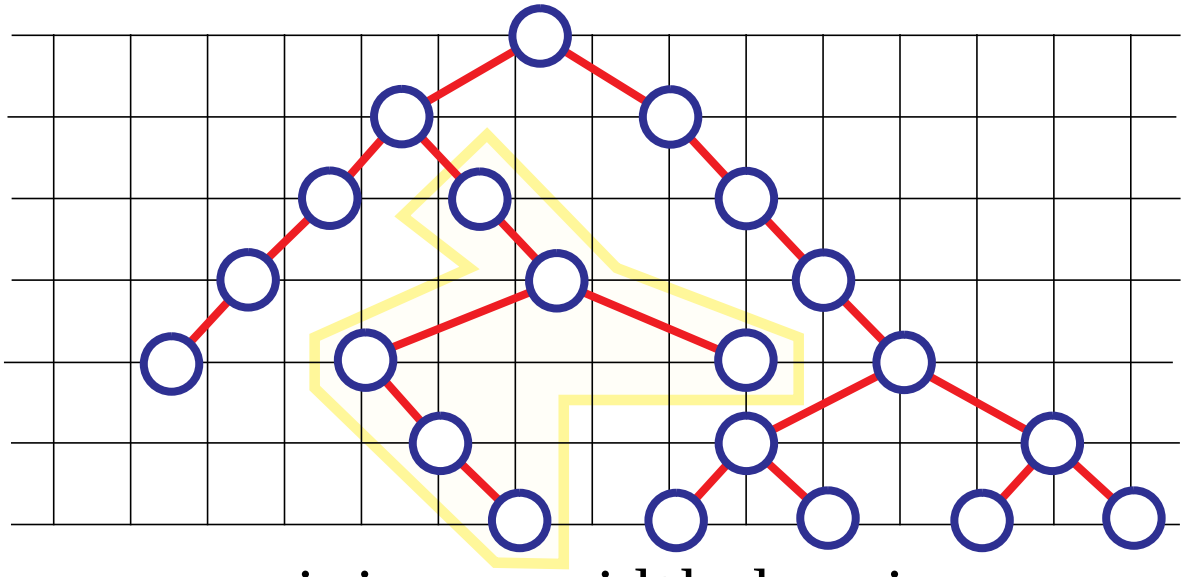


- ***centered*** level drawing
- "small" width
- display of symmetries and of isomorphic subtrees
- can be implemented to run in O(n) time
- can be extended to draw general rooted trees (e.g., root is placed at the average x-coordinate of its children)

Non Optimality of Recursive Tree Drawing Algorithm



drawing constructed by the algorithm

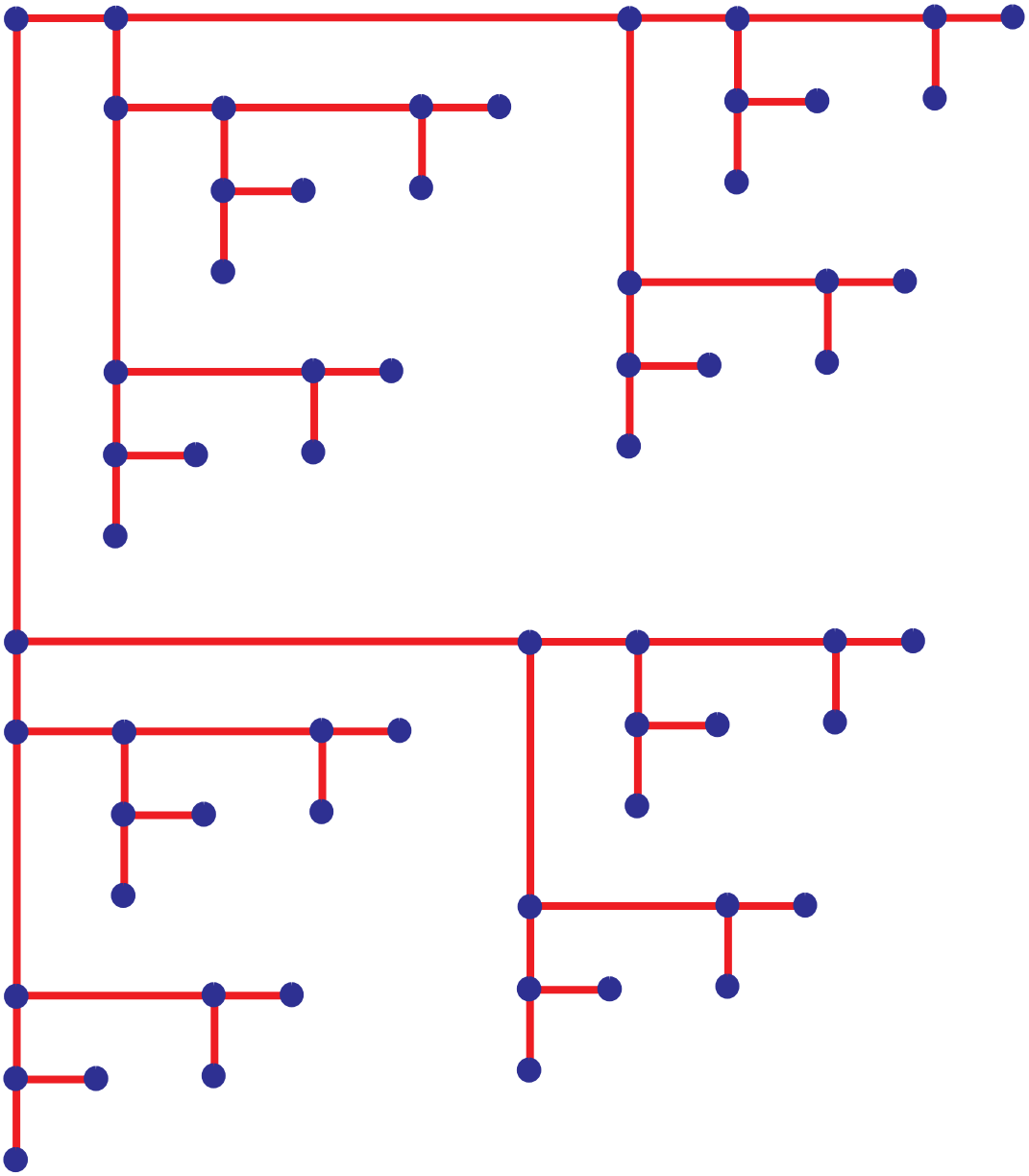


minimum width drawing

- minimizing the width is NP-hard if integer coordinates are required

Area-Efficient Drawings of Trees

- planar straight-line upward grid drawings of ***balanced trees*** with ***O(n) area*** [Crescenzi Di Battista Piperno 92]



Area-Efficient Drawings of Trees

- planar polyline upward grid drawings with ***O(n) area***
  [Garg Goodrich Tamassia 93]

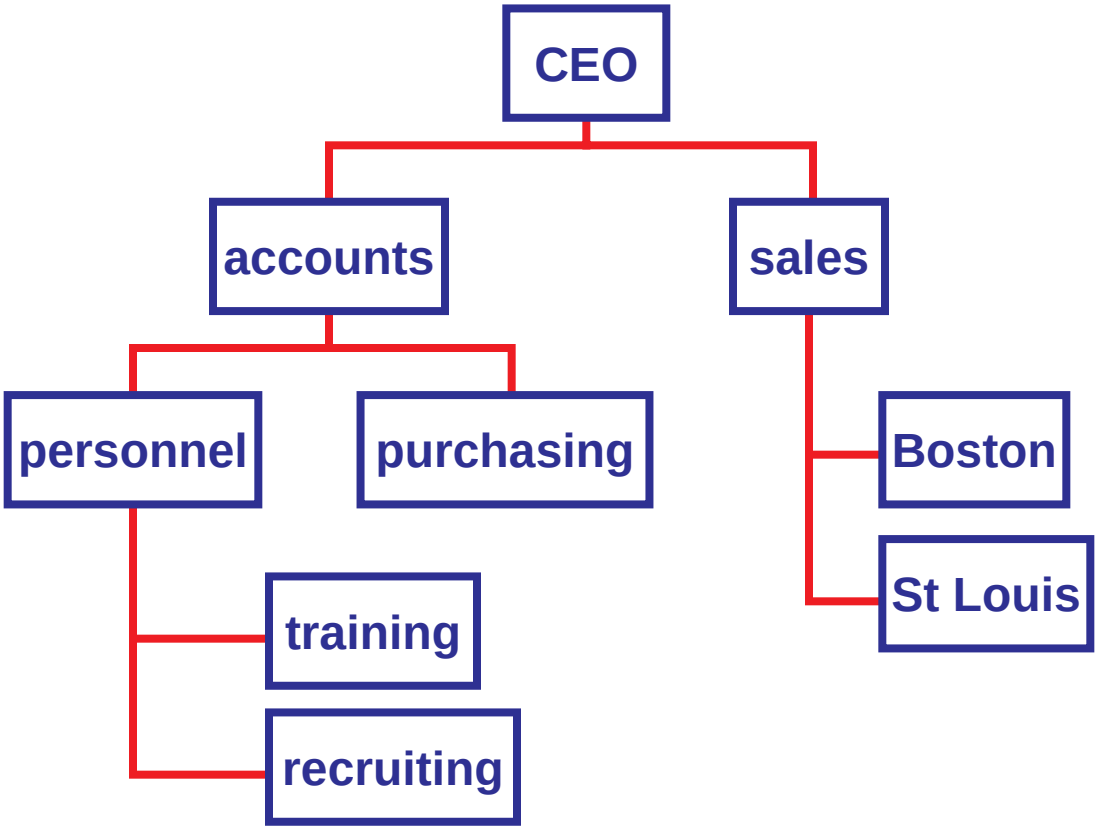
Area Requirement of Planar Drawings of Trees

upward level	$\Theta(n^2)$ [RT 83]
upward polyline	$\Theta(n)$ [GGT 93]
upward straight-line	$\Omega(n)$ $O(n \log n)$ [CDP 92]
upward orthogonal	$\Theta(n \log \log n)$ [GGT 93]
non-upward orthogonal	$\Theta(n)$ [L80, V91]
non-upward leaves-on-hull orthogonal	$\Theta(n \log n)$ [BK 80]

- ***Open Problem***: determine the area requirement of planar upward straight-line drawings of trees

Tip-Over Drawings of Rooted Trees

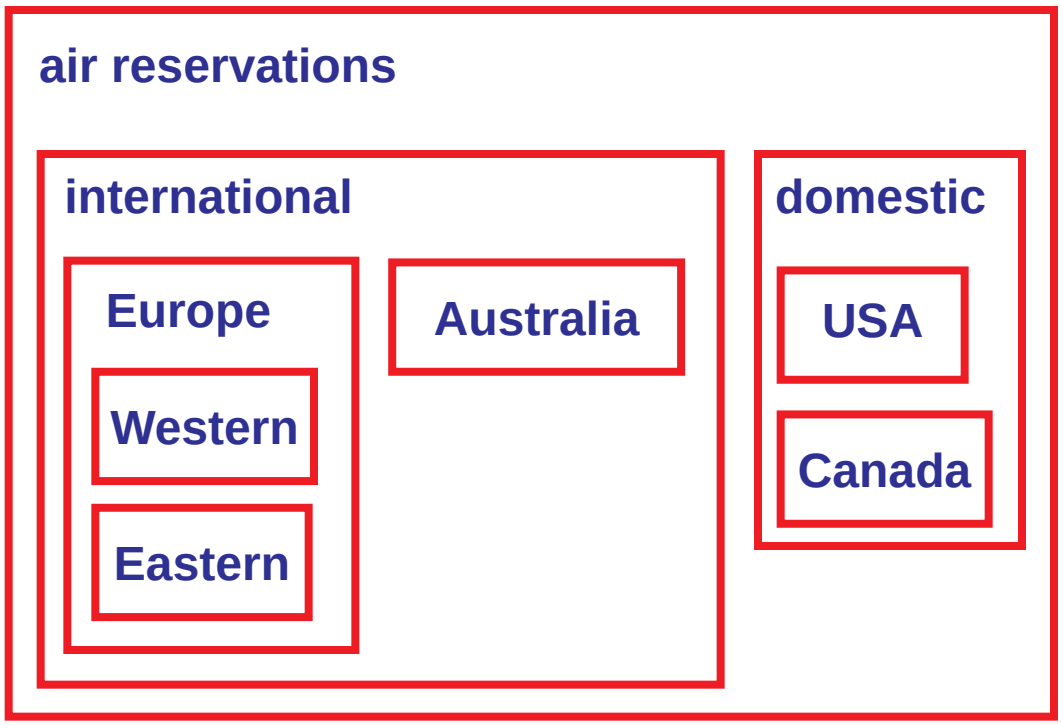
- ***Tip-over drawings*** are upward planar orthogonal drawings such that the children of a node:
  - are arranged either horizontally or vertically
  - share portions of the edges to the parent.



- Widely used in organization charts.
- Allow to better fit the drawing in a

Inclusion Drawings of Rooted Trees

- ***Inclusion drawings*** display the parent-child relationship by the inclusion between isothetic rectangles.



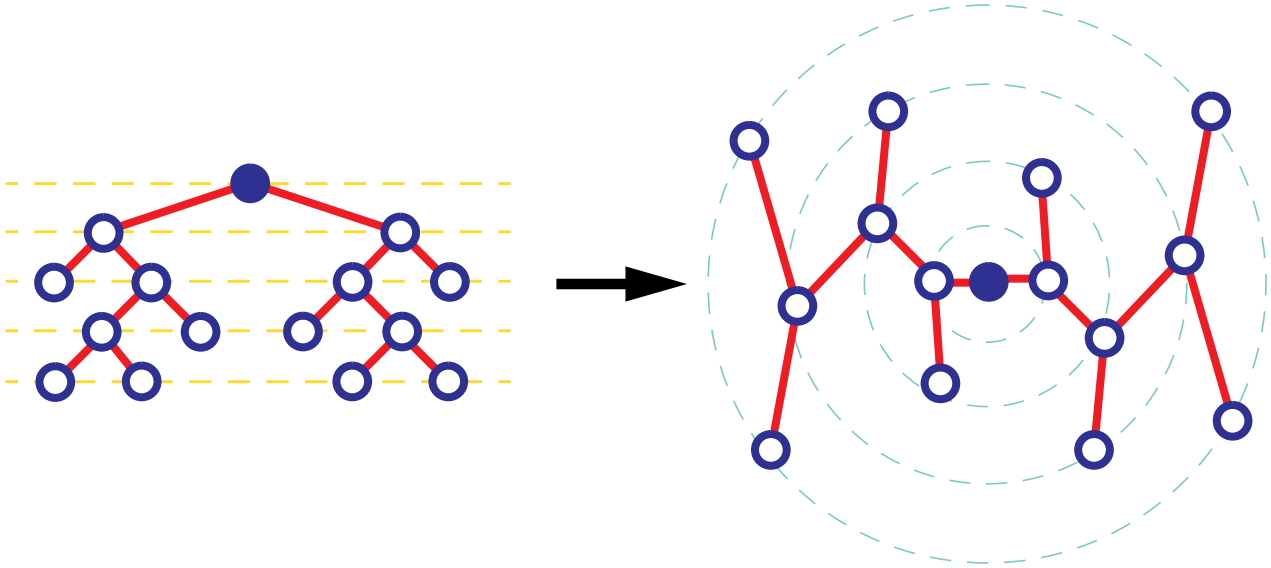
- Closely related to tip-over drawings.
- Used for displaying compound graphs (e.g., the union of a graph and a tree)
- Allow to better fit the drawing in a prescribed region

Area of Tip-Over and Inclusion Drawings

- Eades, Lin and Lin (1992) study of the area requirement of tip-over and inclusion drawings of rooted trees.
- The dimensions of the node labels are given as part of the input.
- ***Minimizing the area*** of the drawing is:
  - ***NP-hard for general trees***
  - computable in ***polynomial time*** for ***balanced trees*** with a ***dynamic programming*** algorithm
- Similar results for the following problems:
  - minimizing the ***perimeter*** of the drawing.
  - minimizing the ***width*** for a given height
  - minimizing the ***height*** for a given width

How to Draw Free Trees

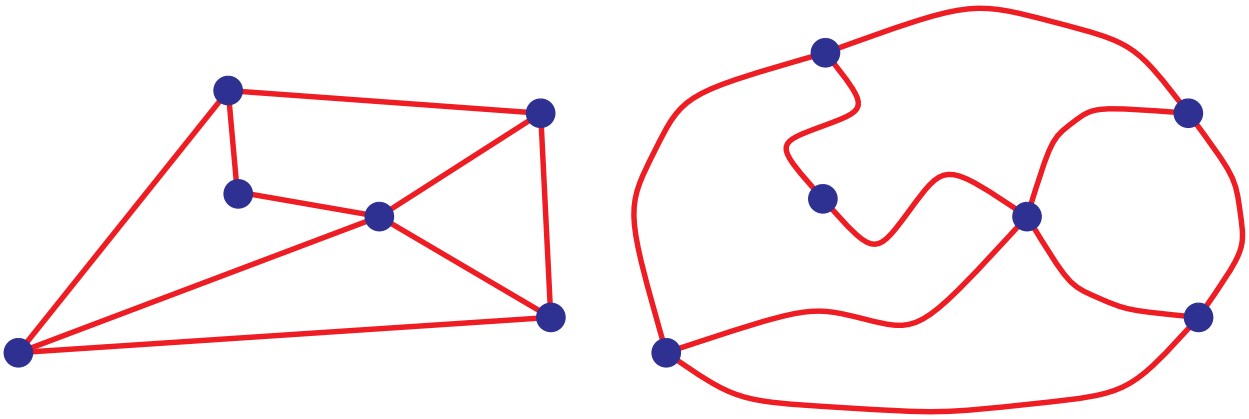
- ***Free trees*** are connected graphs without cycles and do not represent hierarchical relationships (e.g., spanning trees)
- Level drawings of rooted trees yield ***radial drawings*** of free trees:
  - root the free tree T at its ***center*** (node with minmax distance from the leaves), which gives a rooted tree T'
  - construct a level drawing Δ' of T'
  - use a geometric transformation (***cartesian*** → ***polar***) to obtain from Δ' a radial drawing Δ of T



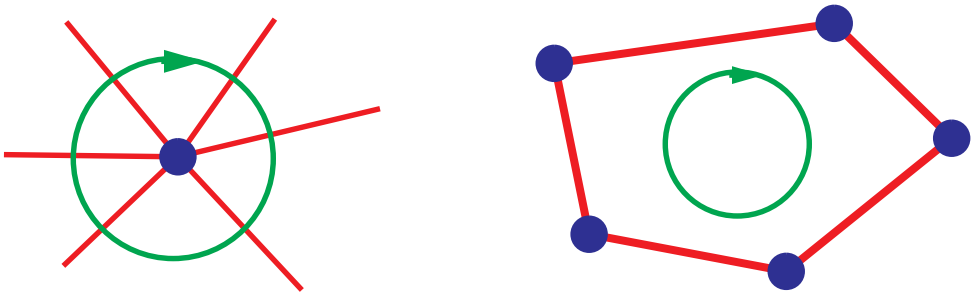
Planar Undirected Graphs

Planar Drawings and Embeddings

- a ***planar embedding*** is a class of topologically equivalent planar drawings



- a planar embedding prescribes
  - the ***star*** of edges around each vertex
  - the ***circuit*** bounding each face



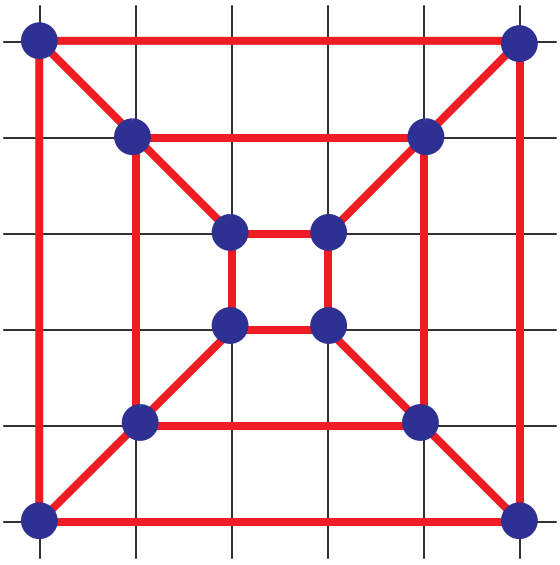
- the number of distinct embeddings is exponential in the worst case
- triconnected planar graphs have a unique embedding

The Complexity of Planarity Testing

- Planarity testing and constructing a planar embedding can be done in ***linear time***:
  - ***depth-first-search***
    [Hopcroft Tarjan 74]
    [de Fraysseix Rosenstiehl 82]
  - ***st-numbering and PQ-trees***
    [Lempel Even Cederbaum 67]
    [Even Tarjan 76]
    [Booth Lueker 76]
    [Chiba Nishizeki Ozawa 85]
- The above methods are ***complicated*** to understand and implement
- ***Open Problem:***
  - devise a ***simple*** and ***efficient*** planarity testing algorithm.

Planar Straight-Line Drawings

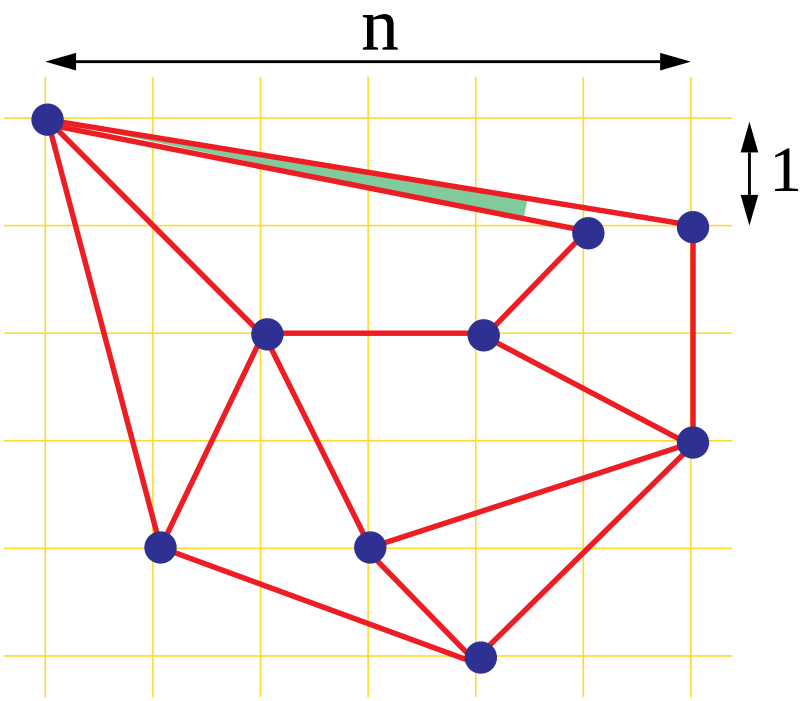
- [Hopcroft Tarjan 74]: planarity testing and constructing a planar embedding can be done in O(n) time
- [Fary 48, Stein 51, Steinitz 34, Wagner 36]: every planar graph admits a planar straight-line drawing



- Planar straight-line drawings may need $\Omega(n^2)$ area
- [de Fraysseix Pach Pollack 88, Schnyder 89, Kant 92]: $O(n^2)$-area planar straight-line grid drawings can be constructed in O(n) time

Planar Straight-Line Drawings: Angular Resolution

- $O(n^2)$-area drawings may have $\rho = O(1/n^2)$



- [Garg Tamassia 94]:
  - ***Upper bound*** on the angular resolution:

$$\rho = O\left(\sqrt{\frac{\log d}{d^3}}\right)$$

  - ***Trade-off*** (area vs. angular resolution):

$$A = \Omega(c^{\rho n})$$

- [Kant 92] Computing the optimal angular resolution is ***NP-hard***.

Planar Straight-Line Drawings: Angular Resolution

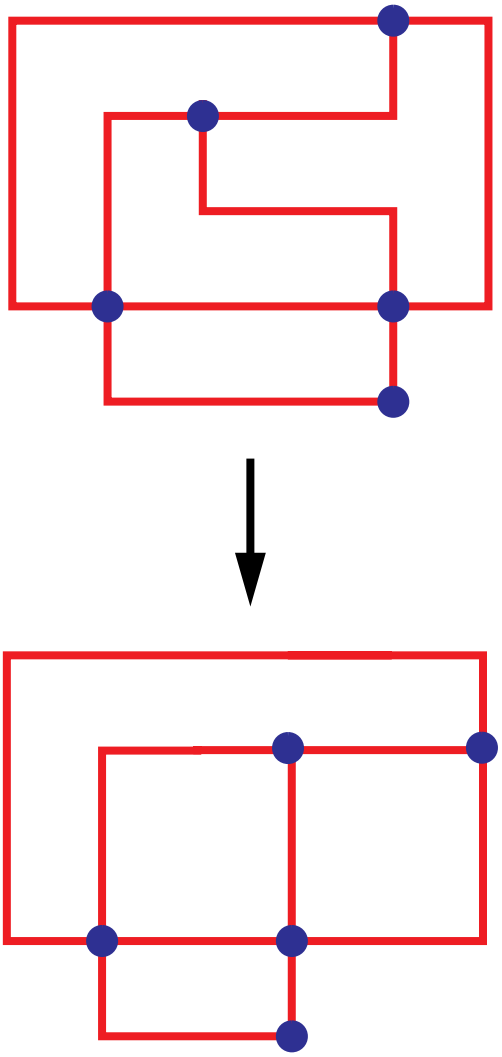
- [Malitz Papakostas 92]: the angular resolution depends on the degree only:

$$\rho = \Omega\left(\frac{1}{7^d}\right)$$

- Good angular resolution can be achieved for special classes of planar graphs:
  - ***outerplanar graphs***, $\rho = O(1/d)$ [Malitz Papakostas 92]
  - ***series-parallel graphs***, $\rho = O(1/d^2)$ [Garg Tamassia 94]
  - ***nested-star graphs***, $\rho = O(1/d^2)$ [Garg Tamassia 94]
- ***Open Problems***:
  - can we achieve $\rho = O(1/d^k)$ (k a small constant) for all planar graphs?
  - can we efficiently compute an ***approximation*** of the optimal angular resolution?

Planar Orthogonal Drawings: Minimization of Bends

- given planar graph of degree ≤ 4, we want to find a planar orthogonal drawing of G with the minimum number of bends

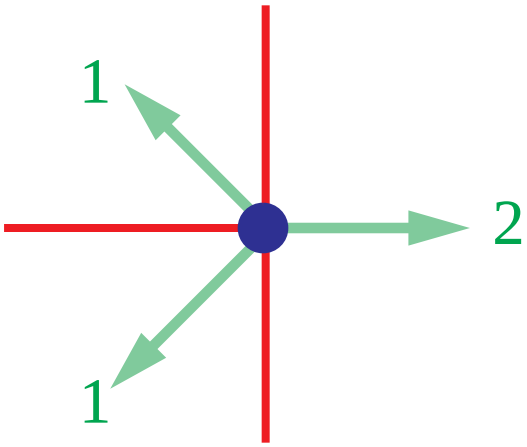


Minimization of Bends in Planar Orthogonal Drawings

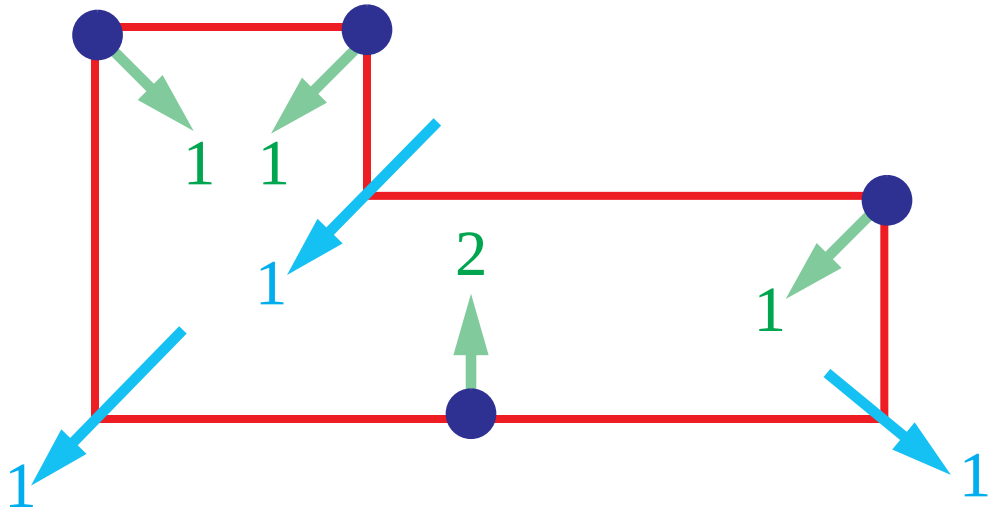
- [Tamassia 87]
  - $O(n^2 \log n)$-time bend minimization for fixed embedding
- [Di Battista Liotta Vargiu 93]
  - polynomial-time bend minimization for degree-3 and series-parallel graphs
- [Tamassia Tollis 89]
  - O(n)-time approximation with O(n) bends
- [Garg Tamassia 93]
  - minimization of bends is NP-hard
  - approximation with $O(opt + n^{1-\varepsilon})$ bends is NP-hard
  - rectilinear planarity testing is NP-complete

Network Flow Model

- a unit of flow is a 90° angle
- a vertex (source) produces 4 units



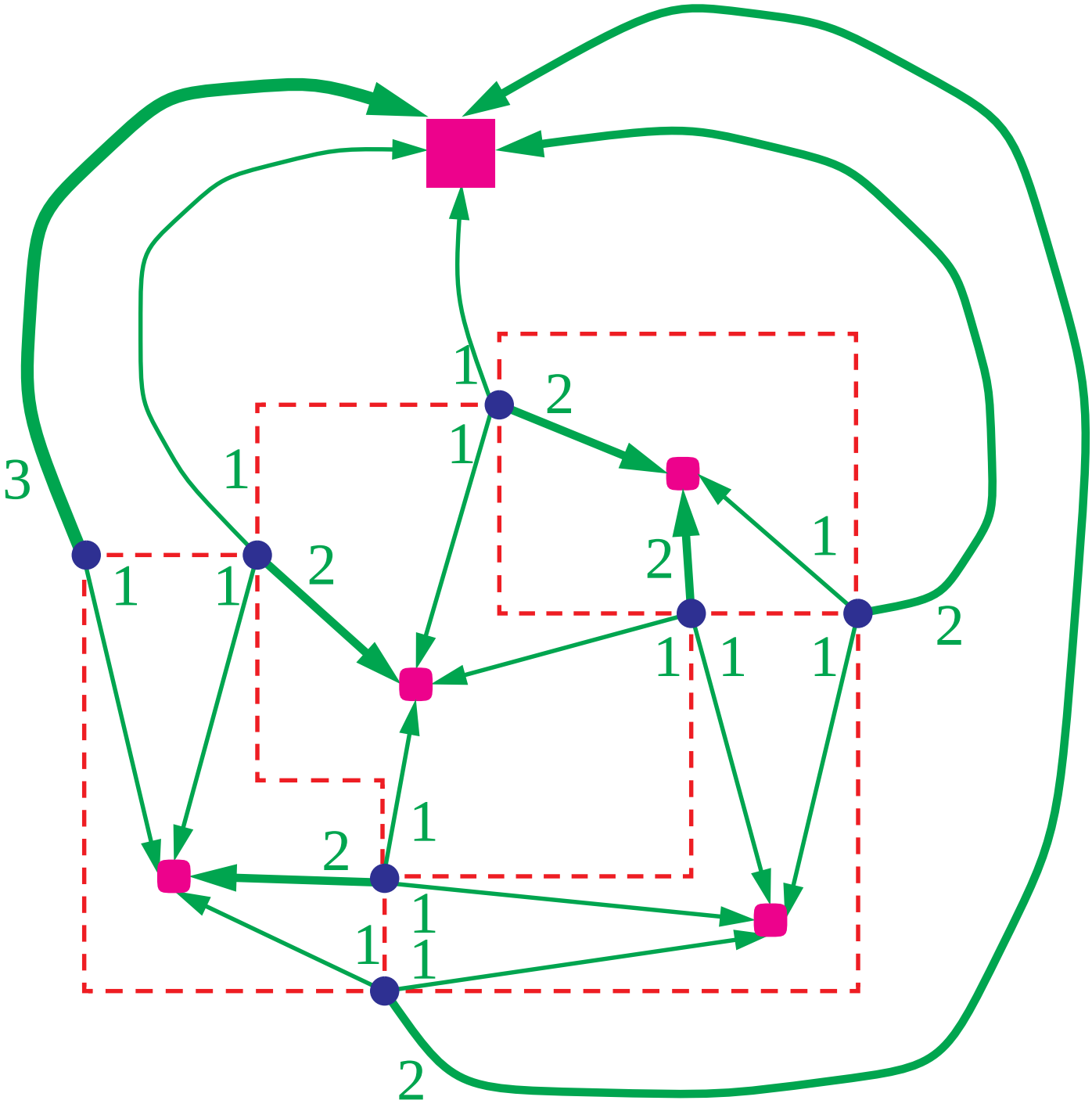
- a face f (sink) consumes 2 deg(f) – 4 units (deg(f) + 4 for the external face)



- Edges transport flow across faces

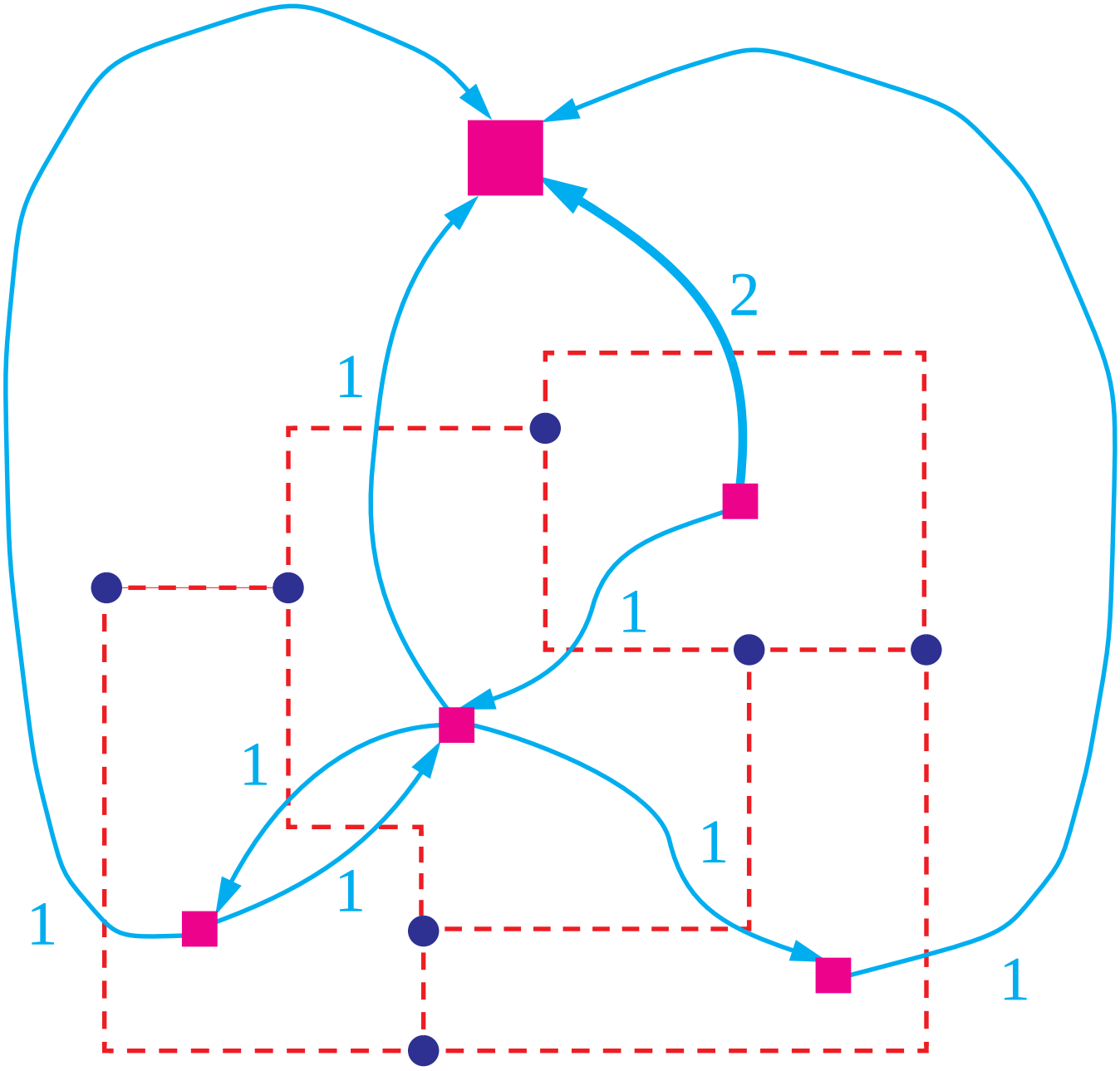
Flow Network

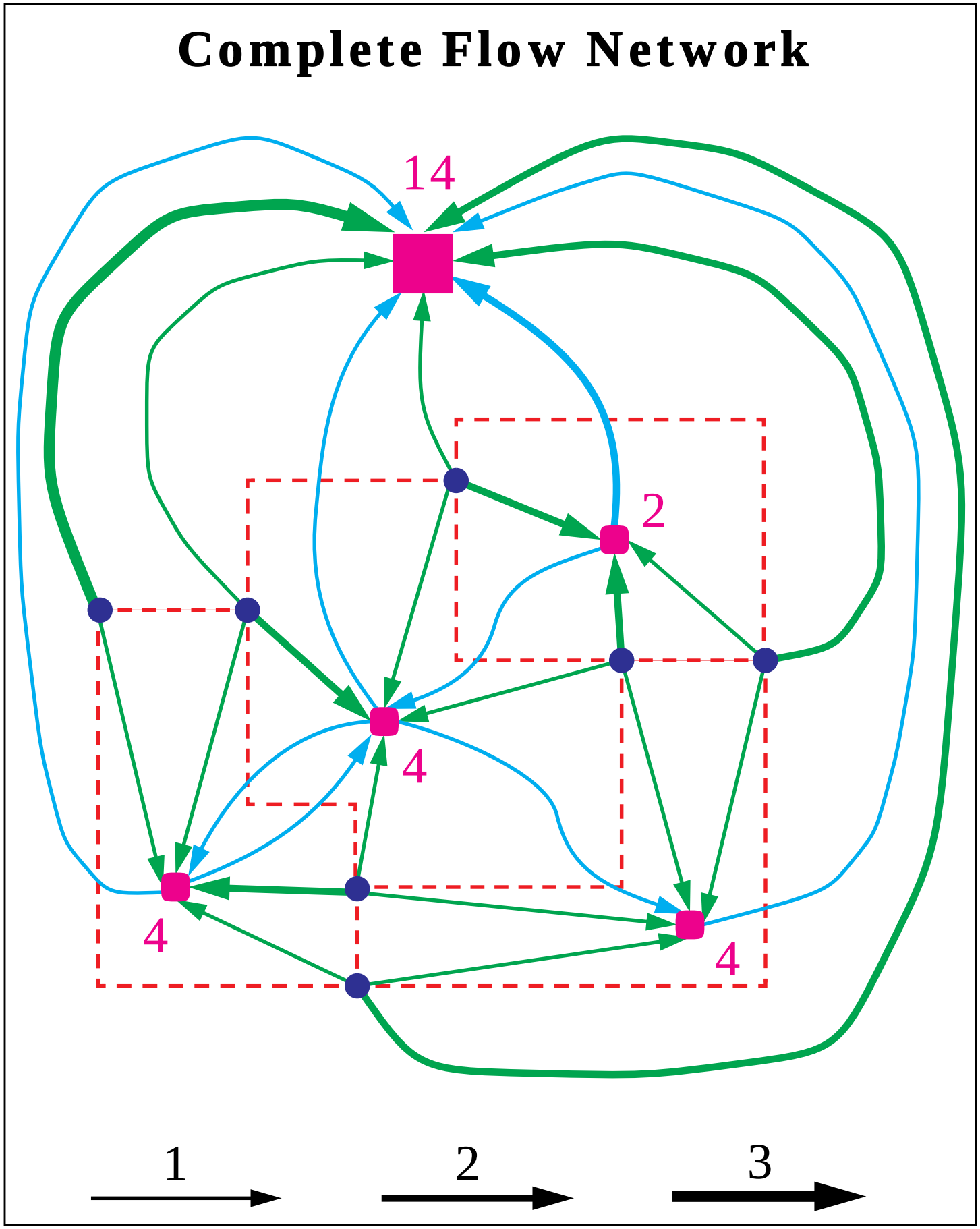
- vertex-face arcs: flow ≥ 1, cost = 0



Flow Network

- face-face arcs: flow ≥ 0, cost = 1




Complete Flow Network
14
2
4
4
4
1
2
3

Correctness of Flow Model

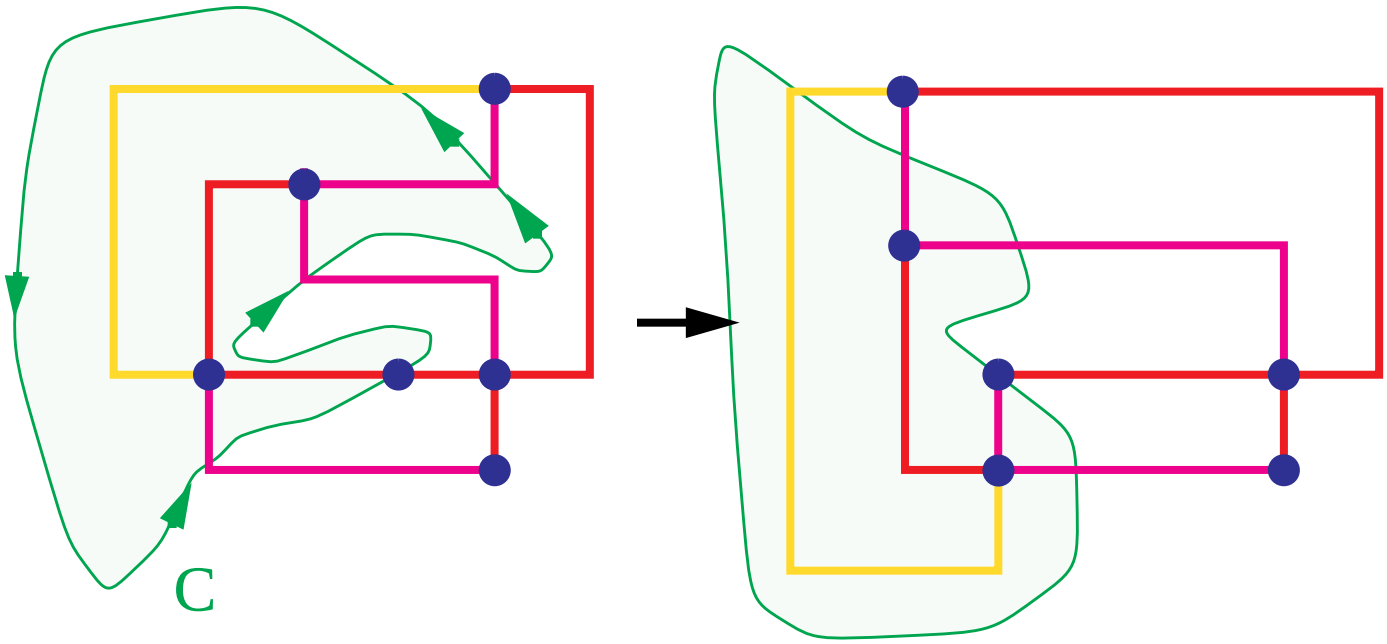
- supply of sources = demand of sinks ↔ Euler's formula
- flow conservation at vertex ↔ ∑ angles around vertex = 360°
- flow conservation at face ↔ (# 90° angles) – (# 270° angles) = 4
- cost of flow ↔ # bends
- flow in N ↔ drawing of G
- minimum cost flow ↔ optimal drawing

Theorem [Tamassia 87] Computing the minimum number of bends for an embedded graph G is equivalent to computing a minimum cost flow in network N, and takes $O(n^2 \log n)$ time

Open Problem: reduce the time complexity of bend minimization.

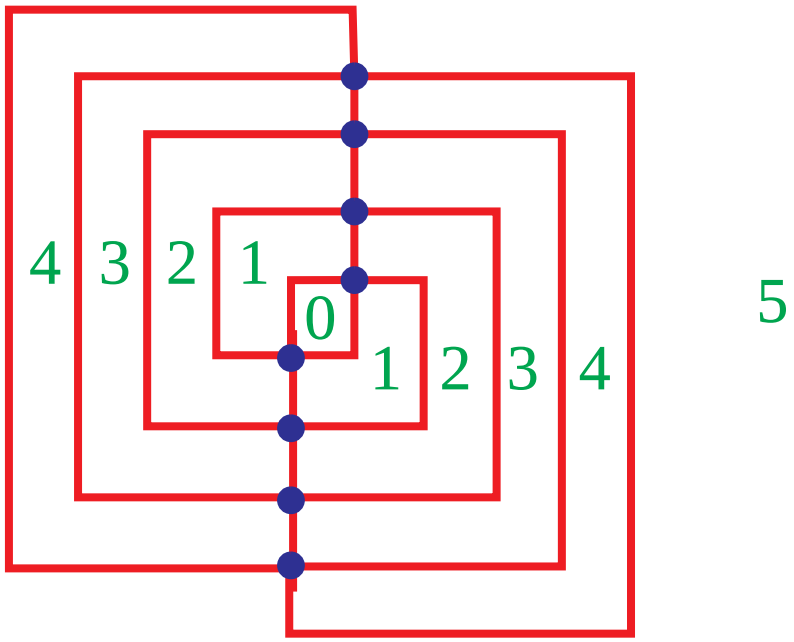
Characterization of Bend-Minimal Drawings

- A drawing has the minimum number of bends if and only if there is no oriented closed curve C such that
  - vertices are intersected by C entering from angles ≥ 180°
  - (# edges crossed by C from 90° or 180°) < (# edges crossed by C from 270°)
- If such a curve exists, "rotating" the portion of the drawing inside C reduces the number of bends



Proving the Optimality of a Drawing

- potential Φ on each face

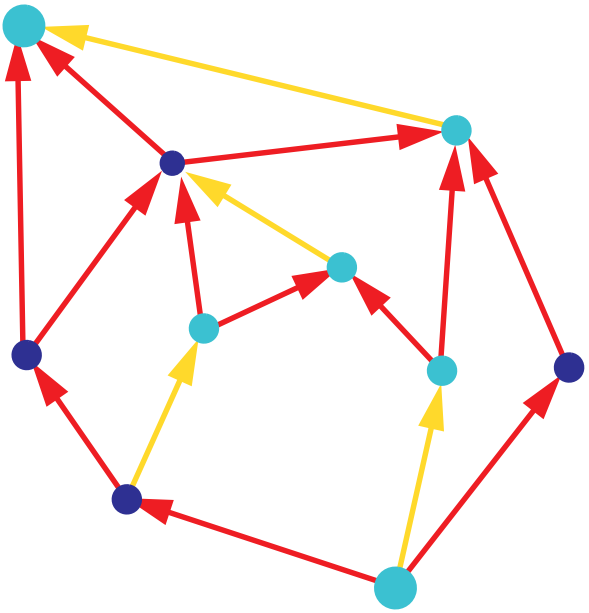


- vertices cannot be traversed by C
- C traverses edge from 270° $\Rightarrow$ $\Delta\Phi_i = -1$
- C traverses edge from 90° $\Rightarrow$ $\Delta\Phi_i = +1$
- bends removed going "inward" and inserted going "outward" $\Delta B_i + \Delta\Phi_i = 0$
- C is a closed curve $\Rightarrow$ $\Sigma i\ \Delta\Phi_i = 0$
- Hence, $\Sigma i\ \Delta B_i = 0$

Planar Directed Graphs

Upward Planarity Testing

- upward planarity testing for ordered sets has the same complexity as for general digraphs (insert dummy vertices on transitive edges)
- [Kelly 87, Di Battista Tamassia 87]: upward planarity is equivalent to subgraph inclusion in a planar st-digraph (planar acyclic digraph with one source and one sink, both on the external face)



- [Kelly 87, Di Battista Tamassia 87]: upward planarity is equivalent to upward straight-line planarity

Complexity of Upward Planarity Testing

- [Bertolazzi Di Battista Liotta Mannino 91]
  - $O(n^2)$-time for fixed embedding
- [Hutton Lubiw 91]
  - $O(n^2)$-time for single-source digraphs
- [Bertolazzi Di Battista Mannino Tamassia 93]
  - $O(n)$-time for single-source digraphs
- [Garg Tamassia 93]
  - NP-complete

Angles in Planar Upward Drawings

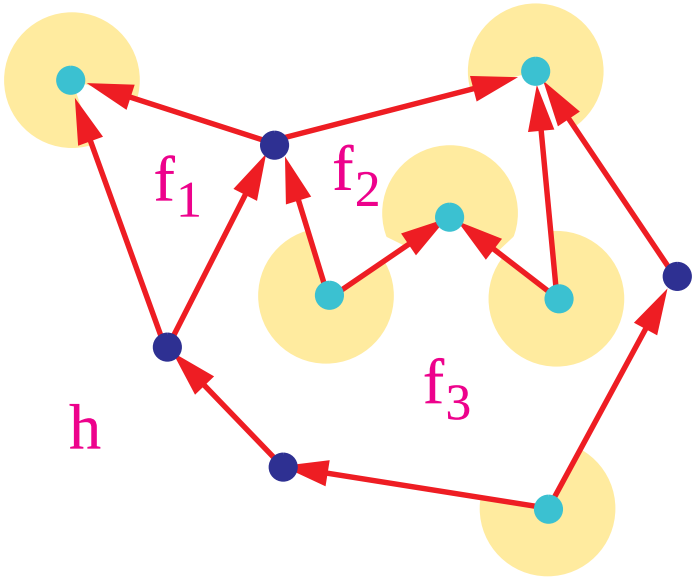
- in any upward drawing the incoming (outgoing) edges of a vertex v appear consecutively in the star of v
- s(f) = (# sources of face f) = (# sinks of face f)
- in a planar straight-line upward drawing
  - each ***pole*** (source or sink of the graph) has exactly one incident ***large angle***
  - an internal face f has s(f) – 1 large angles

$s(f_1) = 1$

$s(f_2) = 2$

$s(f_3) = 3$

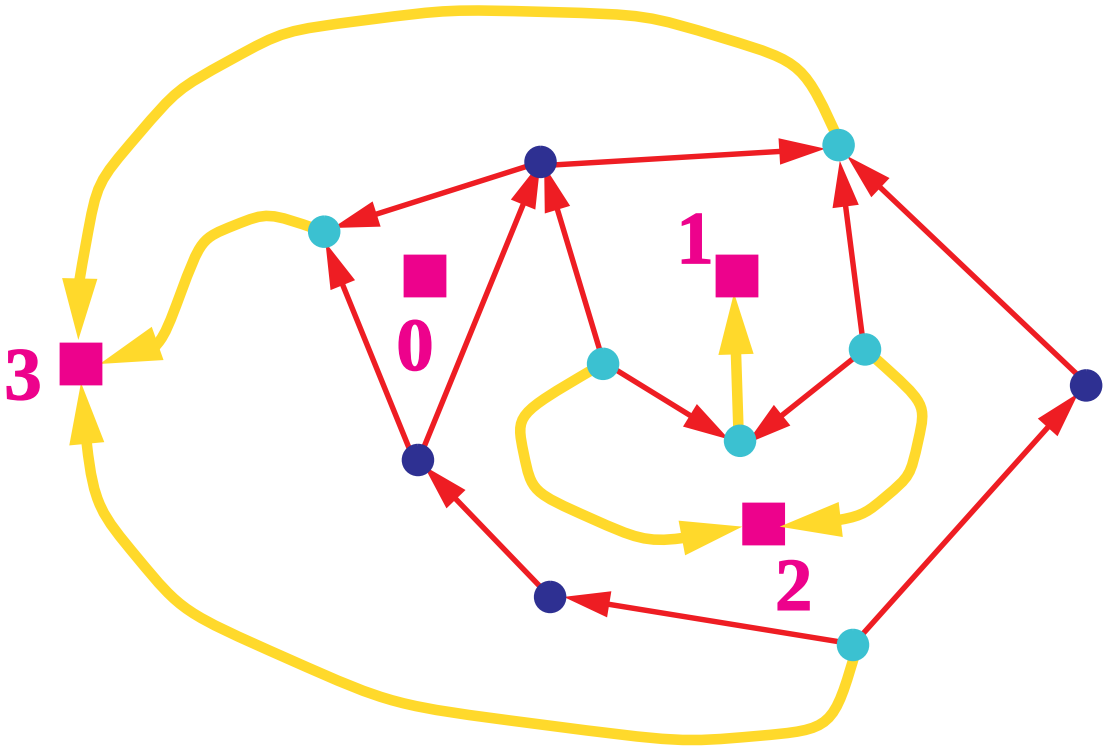
$s(h) = 2$



  - the external face h has s(h) + 1 large angles

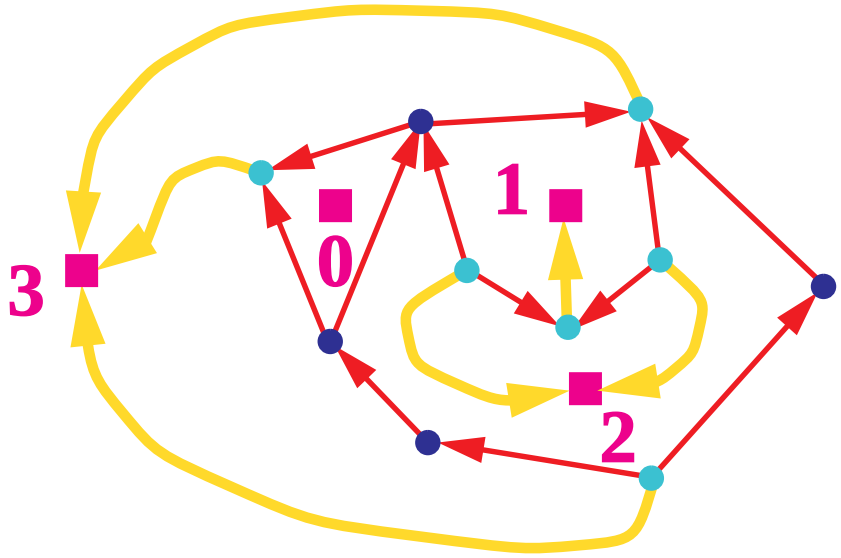
Assignment Model

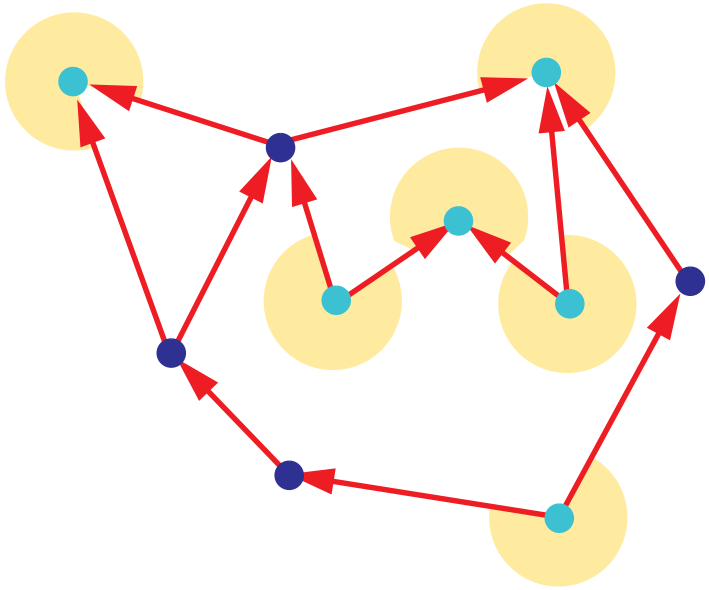
- assignment of poles (sources and sinks) to faces indicating which angles are large
- ***consistent assignment***:
  - each pole v is assigned to exactly one face f incident on v
  - each internal face f has s(f) – 1 poles assigned to it
  - the external face h has s(h) + 1 poles assigned to it



Upward Planarity Testing

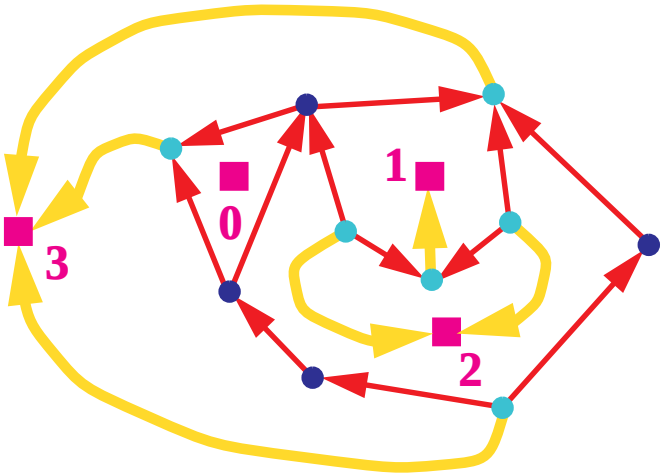
Theorem [Bertolazzi Di Battista 91]
An embedded digraph admits an upward drawing if and only if it admits a consistent assignment of poles to faces

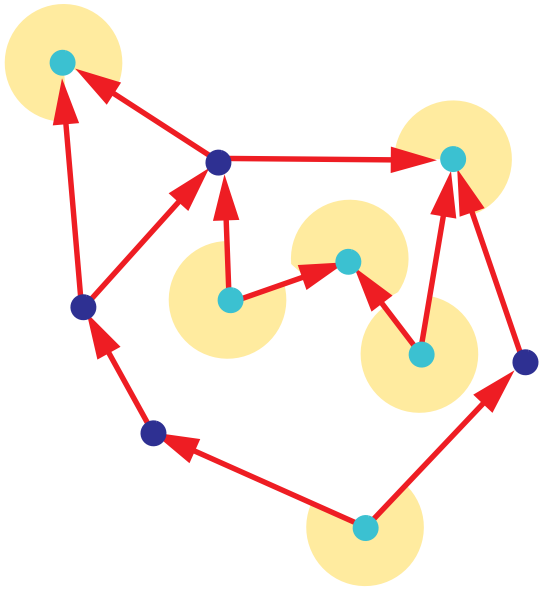


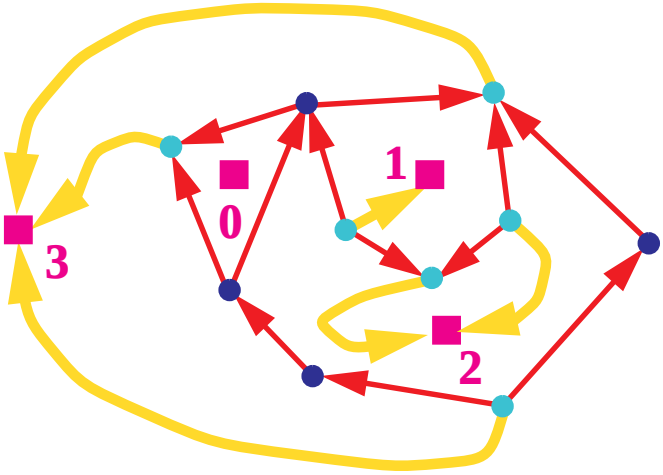


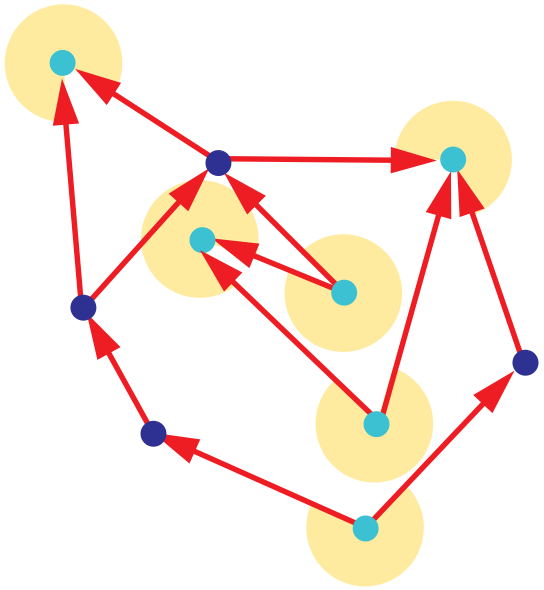
Assignments and Drawings

- distinct consistent assignments yield distinct upward embeddings









Flow Model

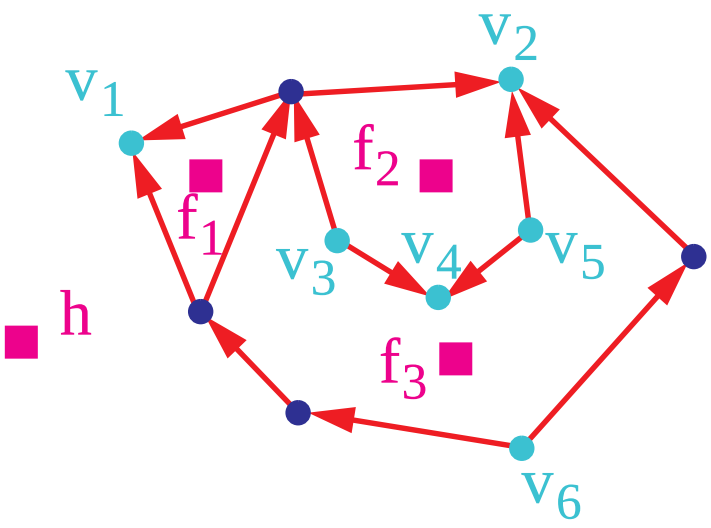
- computing a consistent assignment corresponds to finding a maximum flow in the pole-face incidence digraph, which takes $O(n^2)$ time

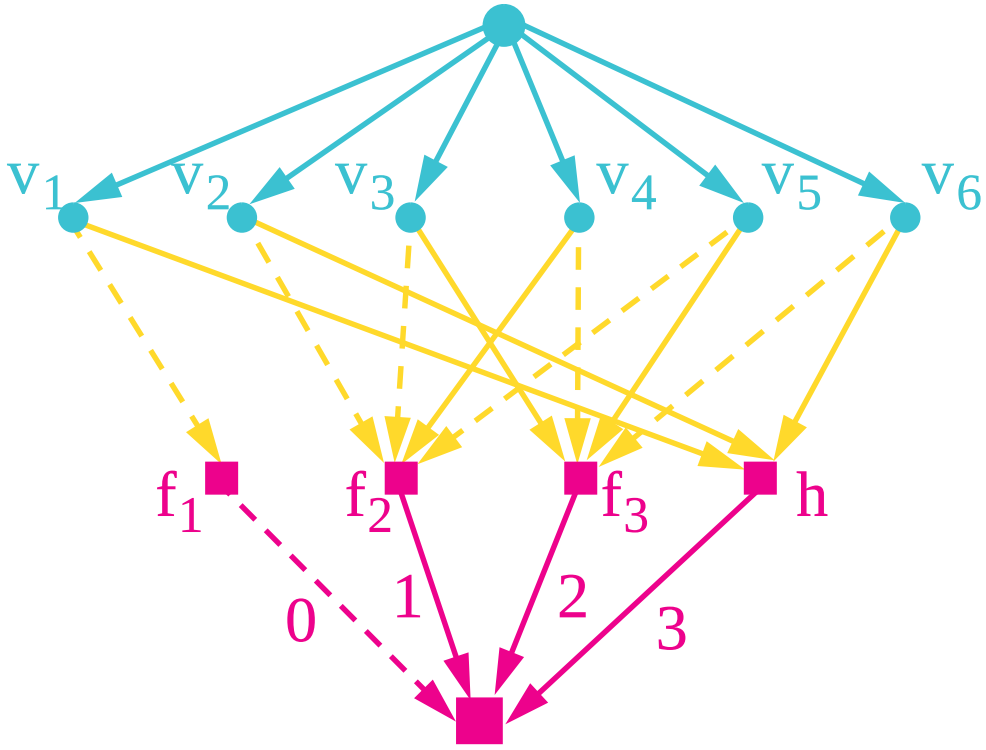
$s(f_1) = 1$

$s(f_2) = 2$

$s(f_3) = 3$

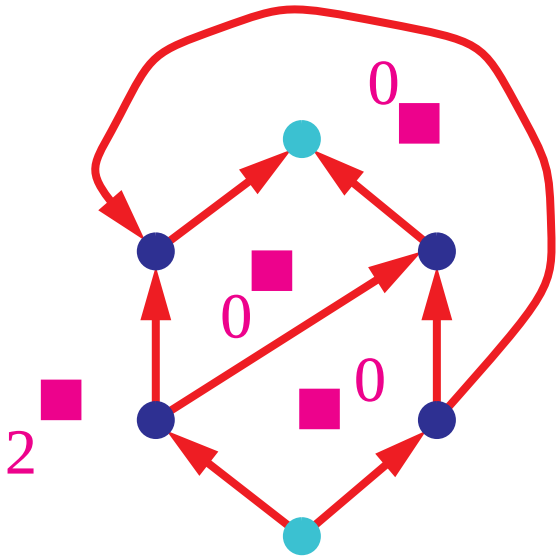
$s(h) = 2$



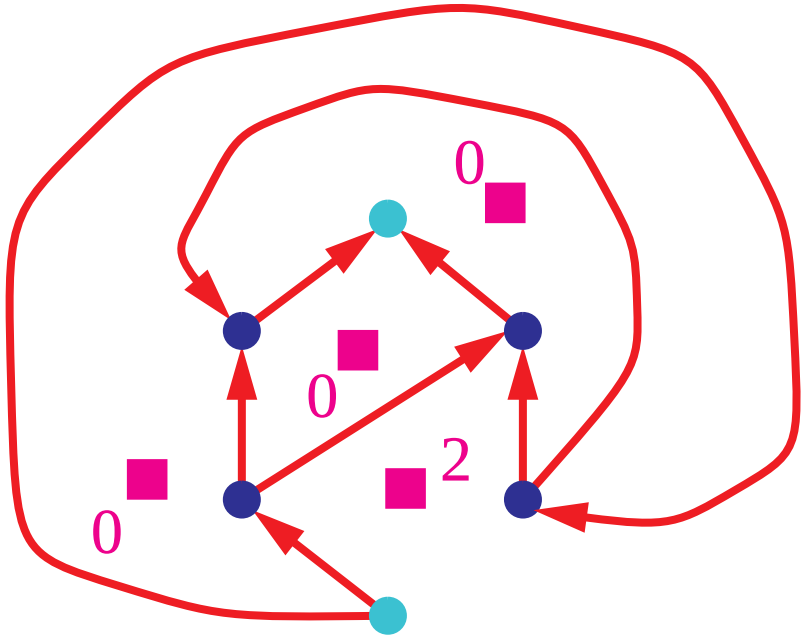


Proving that a Digraph is not Upward Planar

- the embedded digraph below has no consistent assignment ...

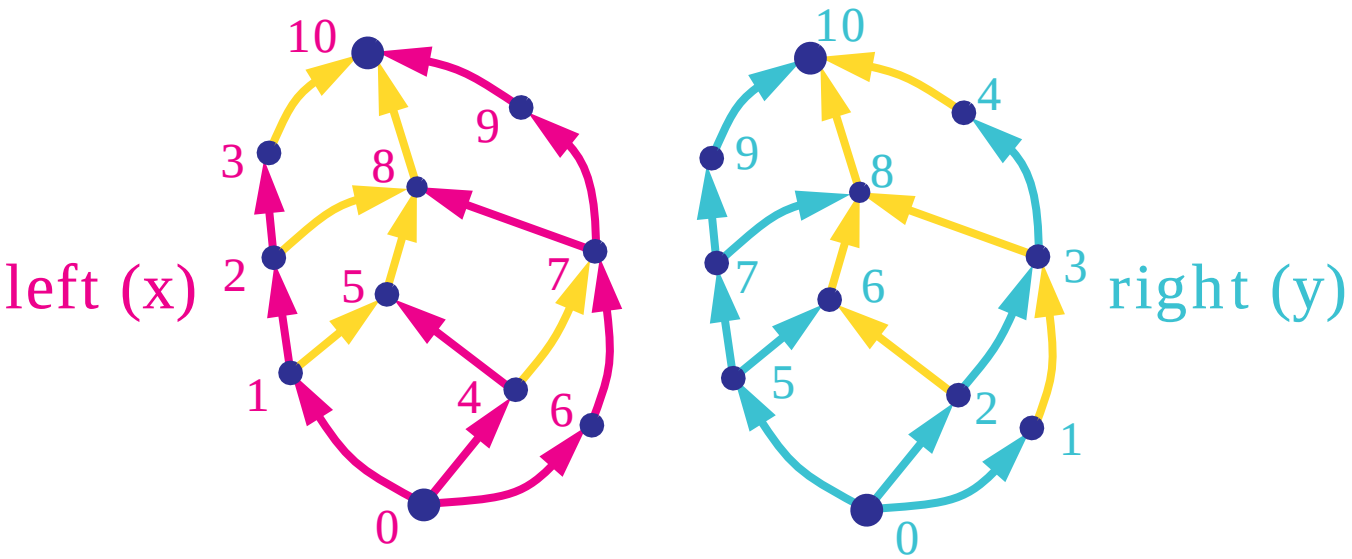


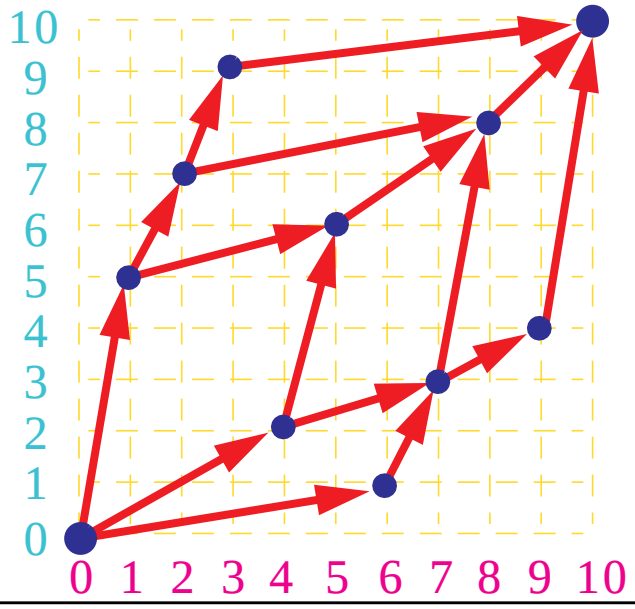
- ... even changing the external face



How to Construct Upward Planar Drawings

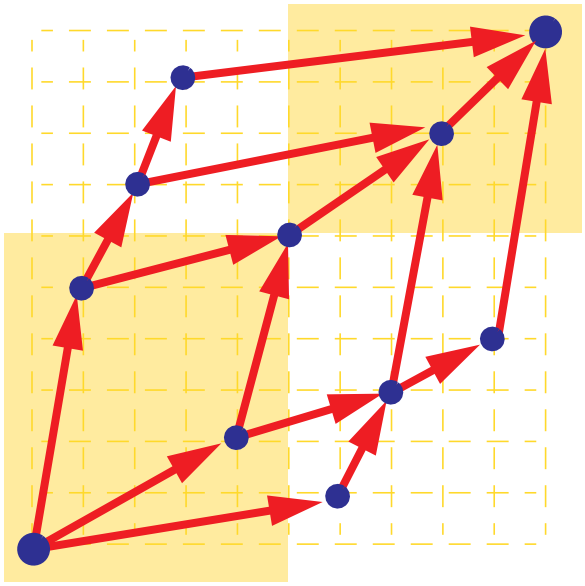
- Since an upward planar digraph is a subgraph of a ***planar st-digraph***, we only need to know how to draw planar st-digraphs
- If G is a planar st-digraph without transitive edges, we can use the ***left***/***right*** numbering method to obtain a ***dominance drawing***:



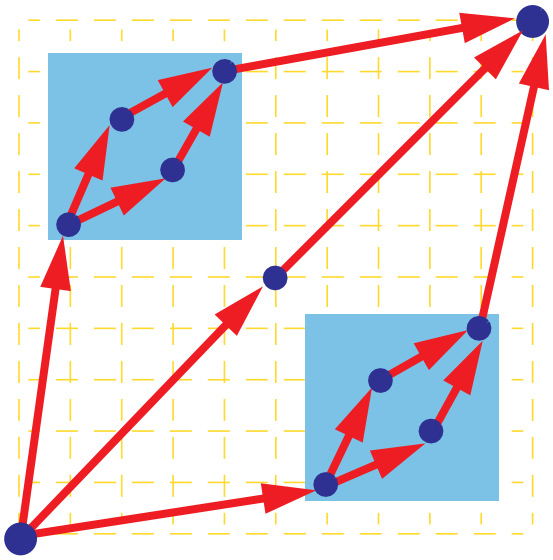


Properties of Dominance Drawings

- ***Upward, planar, straight-line, $O(n^2)$ area***
- The ***transitive closure*** is visualized by the geometric dominace relation

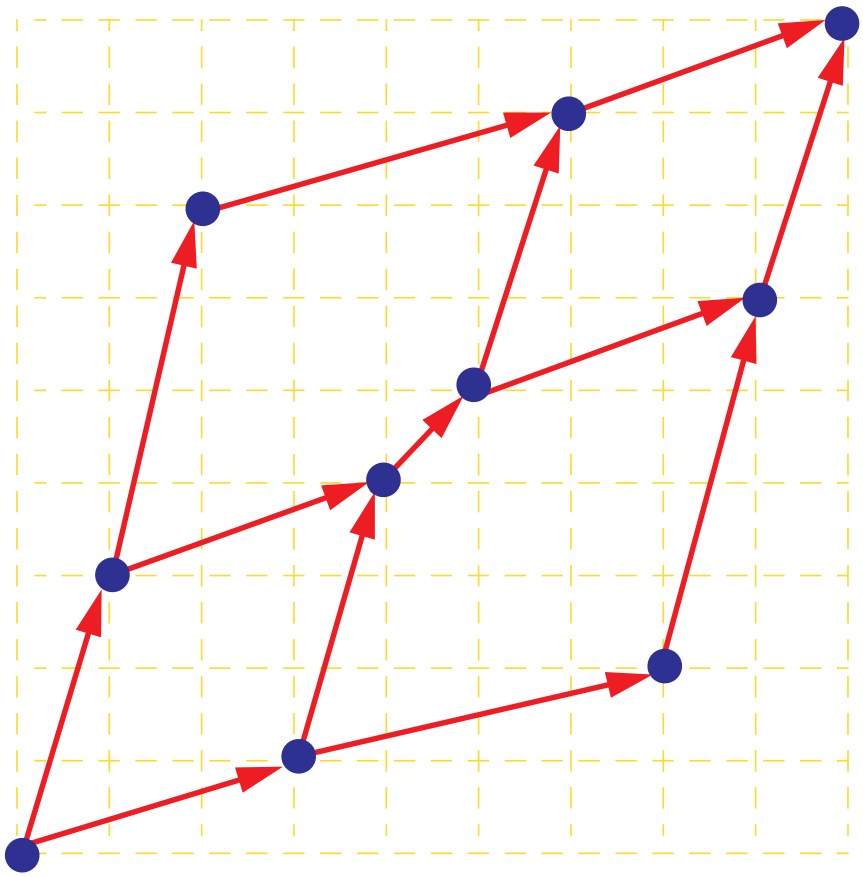


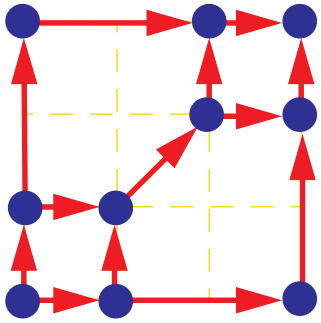
- ***Symmetries*** and ***isomorphisms*** of ***st-components*** are displayed



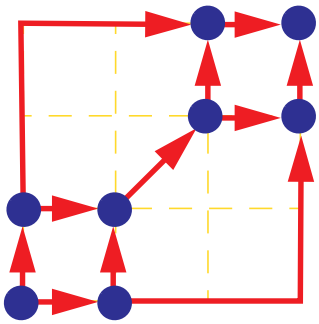
More on Dominance Drawings

- A variation of the left/right numbering yields dominance drawings with ***optimal area***



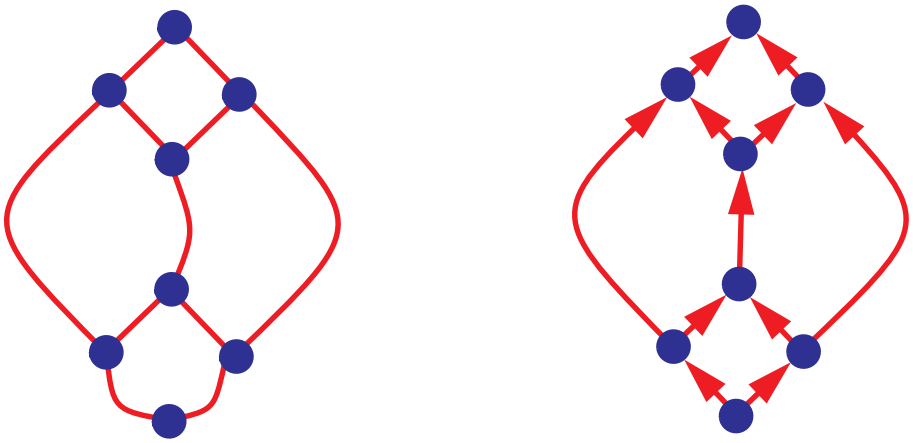


- Dummy vertices are inserted on transitive edges and are displayed as bends (upward planar polyline drawings)

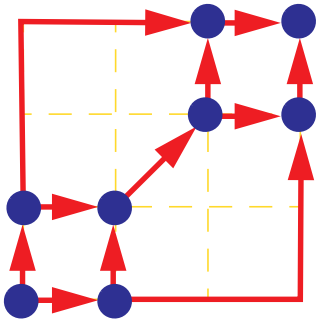


Planar Drawings of Graphs and Digraphs

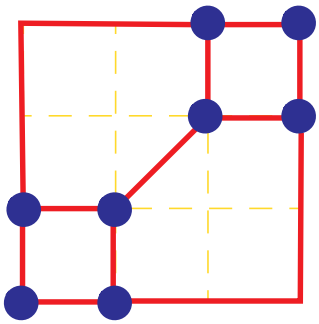
- We can use the techniques for dominance drawings also for undirected planar graphs:
  - orient G into a planar st-digraph G'



  - construct a dominance drawing of G'



  - erase arrows ...



General Undirected Graphs

Algorithmic Strategies for Drawing General Undirected Graphs

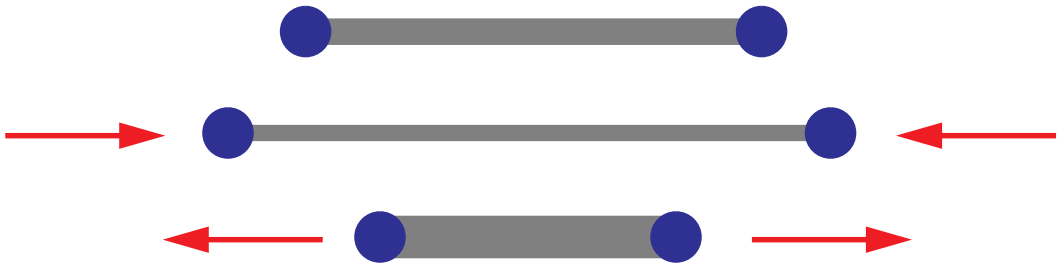
- ***Planarization method***
  - if the graph is nonplanar, ***make it planar***! (by placing dummy vertices at the crossings)
  - use one of the drawing algorithms for planar graphs

  e.g., GIOTTO [Tamassia Batini Di Battista 87]
- ***Orientation method***
  - ***orient*** the graph into a digraph
  - use one the drawing algorithms for digraphs
- ***Force-Directed method***
  - define a ***system of forces*** acting on the vertices and edges
  - find a ***minimum energy state*** (solve differential equations or simulate the evolution of the system)

  e.g., Spring Embedder [Eades 84]

The Spring Embedder
[Eades 1984]

- replace the edges by ***springs*** with unit natural length
- connect nonadjacent vertices with additional springs with infinite natural length
- recall that the springs attract the endpoints when stretched, and repel the endpoints when compressed



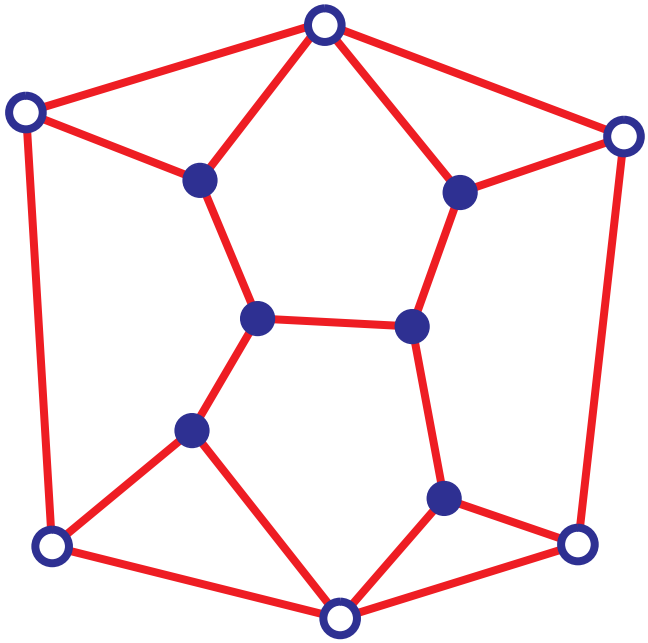
- start with an intital random placement of the vertices
- let the system go ... (assume there is ***friction*** so that a stable minimum energy state is eventually reached)

Other Force-Directed Techniques

- [Kamada Kawai 89]
  - the forces try to place vertices so that their ***geometric distance*** in the drawing is equal to their ***graph-theoretic distance***
  - for each pair of vertices (u,v) use a ***spring*** with natural length dist(u,v)
- [Fruchterman Reingold 90]
  - system of forces similar to that of ***subatomic particles*** and celestial bodies
  - given drawing region acts as wall
  - ***n-body simulation***
- [Davidson Harel 89]
  - ***energy function*** takes into account vertex distribution, edge-lengths, and edge-crossings
  - given drawing region acts as wall
  - ***simulated annealing***

Springs for Planar Graphs

- use springs with natural length 0, and attractive force proportional to the length
- pin down the vertices of the external face to form a convex polygon
- let the system go ...



- the final configuration is a state of minimum energy: $\min \sum_{e} [\text{length}(e)]^2$
- equivalent to the ***barycentric mapping*** [Tutte 60]:

$$\mathbf{p}(v) = 1/\deg(v) \sum_{(v,w)} \mathbf{p}(w)$$

General Directed Graphs

Layering Method for Drawing General Directed Graphs

- ***Layer assignment:*** assign vertices to layers trying to minimize
  - ***edge dilation***
  - ***feedback edges***
- ***Placement:*** arrange vertices on each layer trying to minimize
  - ***crossings***
- ***Routing:*** route edges trying to minimize
  - ***bends***
- ***Fine tuning:*** improve the drawing with local modifications

[Carpano 80]

[Sugiyama Tagawa Toda 81]

[Rowe Messinger et al. 87]

[Gansner North 88]

Systems

Some Graph Drawing Systems

- ***GraphEd***
  (University of Passau, Germany)
  - `ftp.uni-passau.de/pub/local/graphed/`
  - M. Himsolt (`himsolt@fmi.uni-passau.de`)
- ***DAG, DOT, NEATO***
  (AT&T Bell Labs)
  - `research.att.com/dist/drawdag/`
  - S. North (`north@research.att.com`)
- ***Diagram Server***
  (University of Rome)
  - `infokit.dis.uniroma1.it/public/`
  - G. Di Battista
    (`dibattista@iasi.rm.cnr.it`)
- ***Graph Layout Toolkit***
  (Tom Sawyer Software)
  - B. Madden (`bmadden@tomsawyer.com`)

Challenges and Open Problems

- ***3D Graph Drawing***
  - [Cohen Eades Lin Ruskey 94]
  - [Reiss 94]
- ***Dynamic Graph Drawing***
  - [Cohen Di Battista Tamassia Tollis 92]
  - [Hornick Miriyala Tamassia 93]
- ***Aspect Ratio***
  - [Garg Goodrich Tamassia 93]
  - [Chrobak Nakano 94]
- ***Planarization***
  - [Juenger Mutzel 93]
- ***Angular Resolution***
  - [Formann et al 91]
  - [Malitz Papakostas 92]
  - [Garg Tamassia 94]